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The Special and General Theory of Relativity

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Preface

It is called *annus mirabilis*, the year 1905. An unknown clerk in the Bern Patent Office in Switzerland published a succession of four papers in the prestigious German journal *Annalen der Physik*, and the world of physics was changed forever.

During this miraculous year, Albert Einstein revolutionized the field of physics with his special theory of relativity, which along with the work of other scientists in the emerging field of quantum mechanics, such as Niels Bohr, Werner Heisenberg, Wolfgang Pauli, and Erwin Schrödinger, laid the foundation for modern physics. The emergence of this paradigm shift not only brought about new concepts of the universe but also a counterintuitive view of space and time. Now common sense in scientific investigation was blatantly violated, as scientists grappled with new questions about reality.

Accustomed to Newtonian deterministic laws, physicists were now faced with an array of observations at the microscopic level where only probabilities of behavior made sense. Strange things happened with which Newtonian (classical) physics was no longer equipped to deal, shattering the illusion that after electromagnetism, optics, and mechanics were understood at the end of the nineteenth century, there was not much else left for physicists to do but refine their theories and tie up the loose ends. The dual wave-particle nature of matter, the impossibility of defining both momentum and position of a particle simultaneously, the superposition of quantum states, and other quantum phenomena soon exposed the inadequacy of classical physics and called for a radical view.

While quantum mechanics was raising tough questions, Einstein introduced a concept of space and time that deepened our understanding of the behavior of macrostructures such as the stars, the galaxies, and the universe. Of the four papers Einstein submitted, one described the ejection of electrons by photons, now called the photoelectric effect, which helped build the foundation for quantum theory, and which earned him the Nobel prize for physics in 1921. Another paper determined the sizes of molecules from a study of sugar molecules in a water solution, and the number of molecules in a given mass of a substance. The next paper showed how the irregular, zigzagging movements, called Brownian motions, of particles of smoke provide evidence of the existence of molecules and atoms. But the most important is his paper on special relativity. This seminal work from a patent clerk who had to hide his outside investigation in his office drawer whenever he heard footsteps is nothing short of miraculous. Now we know that neither time nor space is absolute, and that we can only speak of time and space in relation to some frame of reference, and that they differ depending on the frames of reference.

Special relativity, as its name implies, is only a special case of the general theory of relativity, and deals with phenomena that occur in inertial reference frames. The general theory investigates behavior in non-inertial frames. We will examine the theory of relativity in its totality.

The tripartite division of this monograph aims at treating the concept of relativity with the thoroughness that will satisfy the curious as well as the student of relativity.

To fix the theory firmly in the reader's mind, I have included copious examples and problems for practice. Some students of physics may find them useful.

As this year marks the one hundredth anniversary of the special theory of relativity, a review of the theory of relativity is a fitting tribute to the genius who dominated science and popular imagination for much of the twentieth century.

Thomas D. Le
31 December 2005

1. Classical Relativity

1.1 Frame of Reference

Einstein's special theory of relativity introduced in 1905 describes the world as it is. Our conception of space and time radically changes by this new insight, and we gain a deeper understanding of physical reality as a result. However, the idea of relativity had existed since Galileo's time, albeit as a constricted one that was adequate enough to account for everyday phenomena within the framework of classical physics. Einstein's contribution was a paradigm shift, not only in that it extended relativity to a wider range of phenomena as it refines the theory but also in that it reveals the universe as vastly different from one conceived with the classical physics paradigm.

The fundamental question of special relativity deals with the differences between the measurements of a single phenomenon made in two different frames of reference that move with a uniform speed relative to each other. These measurements are of space and time since all natural phenomena occur within reference frames that are defined in terms of space and time. Specifically, a **phenomenon** (or an **event**) occurs within the three dimensions of space and one dimension of time, all of which delimit a frame of reference. Although intuitively we may feel that time is somehow different from space, as we know more about Einstein's theory of relativity, we will be convinced that space and time are just part of a continuum, part of the **fabric of spacetime.**

Let us now begin our inquiry with the frame of reference. While riding in a car traveling at a constant speed of 60 miles per hour, you throw a ball up in the air. Where does it land? Straight down into your hand if you do not move. This observation would be identical if you carried out the same experiment standing in your living room. The law of gravity works the same way in either case. Yet you were in two different **frames of reference**, a car moving with uniform speed and a motionless room. There is no experiment you can think of that shows from the behavior of the tossed ball whether you are in a moving car or in a room at rest. We call them **inertial reference frames** because they are either at rest or moving at a constant velocity.

Now think of an observer standing by the roadside while your car was driving past. How did she see the ball drop? To her the ball did not fall down vertically as you had experienced. She saw it fall forward with the velocity of the car in a parabolic curve. The parabolic path of the ball is in accordance with the laws of mechanics. Clearly the ball trajectory differs depending on the frames of reference again, i.e., whether the observer is moving or stationary. However, the stationary observer is in a different inertial frame than the passenger in the car. The observer's reference frame is the earth. Although the earth revolves around its axis and orbits the sun, for practical purposes and for the time being, we can regard its acceleration and rotation as negligible, and the earth as an inertial frame. This shows that **motion is not absolute but relative to the reference frame in which it occurs**. There is equivalence of the laws of mechanics (e.g., law of inertia, of universal gravitation, and so forth) in different inertial frames of reference. And no inertial frame is "preferred" by the laws of mechanics over any other inertial frame since they are all equivalent. You could indifferently say, in the example above, that the car is moving and the earth is at rest, or the car is at rest and the earth is moving, which is what the passenger in the car perceives anyway.

The ball-throwing experiment leads to the same result whether you are traveling in a train, on a boat, or in an airplane moving at a constant (non-accelerated) velocity. The laws of mechanics apply equally to the passenger as well as to the observer fixed on Earth. For example, when a flight attendant in an airplane pours coffee, it fills the cup in the same way that it does when you pour it in your kitchen. Everything in an airplane flying at a uniform speed v' of 1000 km/h obeys the same laws of mechanics as it does on earth. To an observer in the stationary reference frame of the earth, a flight attendant who walks toward the front of the airplane with the uniform speed u' of 3 km/h, adds her speed to that of the airplane. Hence, to an observer on earth, she travels at: $u = u' + v'$, or u = 3 km/h + 1000 km/h = 1003 km/h, although in her own reference frame of the airplane, she walks 3 km/h. In general, if a reference frame S' (e.g., the airplane) moves with a velocity of v' with respect to another reference frame S (e.g., the earth), then the velocity u of an object relative to the frame S is equal to the sum of the two velocities, v' and u', the latter being the velocity of the object in the moving reference frame S'. Of course, if the airplane accelerates as at takeoff or flies through a disturbance, it is no longer an inertial reference frame, and the laws of mechanics do not apply in the same way as before.

Let us take the example of the airplane (which we call reference frame S') traveling with a uniform (i.e., non-accelerated) velocity of 1000 km/h again. All objects in the airplane are traveling with the same velocity relative to the earth. Yet when a passenger in her seat picks up a book to read, they are both at rest relative to the airplane. When she lets go of the book, it falls straight down just as it does in her living room, in accordance with the law of gravity. There is nothing inside the airplane, other than the rumbling of the engines and the occasional display on the monitor in the seat in front of her, that tells her she is traveling at 1000 km/h. The book she is reading travels with the same velocity as she is, so to her it is stationary. To an earth-bound observer (which we call reference frame *S*) the book is not motionless. Who is correct, the passenger or the observer? Both are correct. The difference in their experience comes from the frames of reference in which they find themselves. The passenger and her book are at rest with respect to the reference frame of the plane and are moving at 1000 km/h with respect to the reference frame of the earth. Or you can even correctly say that the flying airplane is at rest and the earth is moving. Any airplane traveler thinks this is the case. The classical laws of mechanics, such as the law of universal gravitation, apply equally in both reference frames. Again, there is no one inertial frame that is "better" or preferred over any other inertial frame.

Now suppose that you are in a cabin of a ship moving in a straight line with constant speed. If you drop a book from the ceiling, it falls straight down, in exactly the same way it would in your house. Not only that, it accelerates at 9.8 m/s^2 as it does on shore. Since your reference frame is an inertial frame, according to Galileo, the uniform movement of the reference frame has no observable effects. Newton's first law of motion (the law of inertia) and second law (the time rate-of-change of momentum) apply in this case. In fact, all laws of mechanics apply as well.

These examples and countless others like them show that there is **no such thing as absolute motion**. There is motion only relative to a given frame of reference. Galileo and Newton knew this and called it **relativity.** (See Section 1.2. below.)

As an illustration, a reference frame *S* (the unprimed frame) is represented by a set of coordinates $\{x, y, z, t\}$, called Cartesian coordinates, the first three of which are graphically represented by three lines that are perpendicular to one another, with the observer normally placed at their common origin *O*, as in Figure 1-1 below, where *x*, *y*, and *z* are spatial dimensions, and *t* is the time dimension. The time dimension *t* does not have a graphical representation. To measure an event (an event occurs in some reference frame) we must know not only *where* it occurred but also *when* it did. Hence a fourth dimension is needed. Likewise, a reference frame *S'* (the primed frame) is described by the set of coordinates $\{x', y', z', t'\}$.

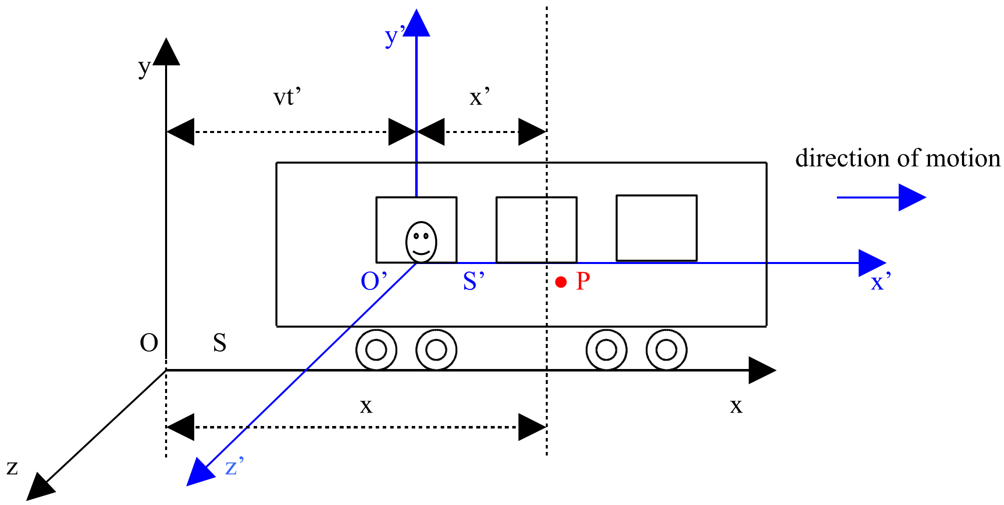


Figure 1-1. Frames of reference

Figure 1-1 illustrates two reference frames: The (unprimed) frame *S* is at rest and the (primed) frame *S'* is in motion (We normally make *S'* the moving frame.). We can consider *S'* as the inertial frame moving with a constant velocity *v* for time *t'* in the *x* direction, for example, a passenger in a train at *O'*. In the frame *S'* the passenger at *O'* sits a distance *x'* from the back of the seat just in front of her at *P*. The train is moving away from the stationary observer at *O* (not shown) on the ground in the frame *S*. The distance *x* from the ground observer to the seat in front of the onboard passenger, which is equal to the distance $vt' + x'$, will increase as the train moves away with velocity *v*. Keep this in mind for the velocity addition discussed in Section 1.2.1 below.

By now, we can see, when speaking of space and time, that everything exists in some frame of reference. Hence, the most important factor in solving relativity problems is to identify the reference frame.

1.2 The Galilean-Newtonian Relativity Principle

The concept of relative motion was known to Galileo and Newton (sometimes called **Galilean** or **Newtonian relativity**) in the seventeenth century. According to this principle, if the laws of mechanics apply in one inertial reference frame, they also apply in any other inertial reference frame moving with constant velocity relative to the first one, so that it is impossible to tell whether an inertial frame is at rest or in motion. In other words, all inertial frames are equivalent and there is no preferred inertial frame. You can verify this with your own experience. While sitting in a train ready to leave the station, you observe the train on the next track moving. For a while you are not sure if your train or the other is pulling out of the station. You can tell only by looking out the window using the background scene as reference point. The same can be said of an observer in the other train. Likewise, while looking out the window of a standing airplane with its engines on, if you see another plane taxiing nearby, you cannot at first tell which plane moves without referring to a fixed object on the ground.

In classical physics, the **Galilean** or **Newtonian relativity principle** holds that all Newtonian laws of mechanics behave the same way in all inertial frames. This is the basic assumption of classical mechanics. And it conforms to common sense as we do not expect these laws to differ when we change places. Indeed, Newton's laws of motion can perfectly account for motions of macrostructures at speeds far less than the speed of light.

However, the Galilean relativity principle failed to apply to Maxwell's theory of electromagnetism, which had been quite successful in describing electromagnetic phenomena, of which light was believed to be one. While the electromagnetic equations predicted the speed of light to be $c = 3 \times 10^8$ m/s, the Galilean transformations allowed a moving body to exceed the speed of light (See Section 1.2.1). This contradiction and studies of motions with velocities approaching that of light soon revealed the inadequacies of Newton's laws and the Galilean relativity principle. The remedy comes from Einstein's special theory of relativity, which can account for objects moving at any speed from zero up to c, thus making the Galilean relativity principle a special case.

1.2.1 Galilean Velocity Addition

Common sense tells us that if a moving person throws a ball forward, the speed of the ball is equal to the person's speed plus the ball's speed. This addition of velocity is a natural result observed in our daily experience, and we never think twice about its consequences for our intuition agrees with the physics of the phenomenon. Nothing we experience contradicts this observation. We can formalize the observation with an example.

Suppose a passenger walks at the speed u' of 3 m/s toward the front of a train traveling at a constant speed v of 30 m/s. What is the speed of the passenger as measured by an observer standing on the platform?

Taking the train as the S' frame and the observer on the platform as the S frame, we have:

$$u = u' + v = 3 \text{ m/s} + 30 \text{ m/s} = 33 \text{ m/s}$$

where u is the passenger's speed relative to the earth (i.e., the stationary observer), u' is the speed of the passenger as measured in the S frame, and v' the speed of the train. This equation represents the **Galilean velocity addition.** As far as the train passenger is concerned, he moves only 3m/s.

Now, if the same passenger walks toward the rear of the train with the same speed, he is moving in the direction opposite to that of the train, or $u' = -3$ m/s. More formally, if the train moves along the x-axis in the x-direction, then the passenger moves in the $-x$-direction. Substituting this value of u' in the formula, we obtain the passenger's speed with respect to the stationary observer in S:

$$u = u' + v = -3 \text{ m/s} + 30 \text{ m/s} = 27 \text{ m/s}.$$

Again, as far as the train passenger is concerned, he moves only 3 m/s. Though both passenger and observer observed the same phenomenon, they arrived at different conclusions. Yet they are both right because this relativity is the nature of the physical world.

Note that our example illustrates everyday speeds, which are far less than the speed of light, or $v << c$. And since we never experience any speeds close to c in daily life, we never suspect anything remiss about classical relativity. The addition of velocity seems to be working correctly.

To see how this velocity addition within the Galilean (classical) relativity does not work at a speed close to the speed of light, consider a Gedanken (thought) experiment. Instead of a passenger in a train, think of an astronaut traveling in a spaceship with the speed close to that of light, $v = 0.90\ c$, and he shoots a rocket forward in the direction of motion of the spaceship at $u' = 0.15\ c$ with respect to the spaceship. What is the rocket's speed relative to an observer on the earth?

Applying the Galilean velocity addition above, we get:

$$\boldsymbol{u = u' + v} \qquad \textbf{(1)}$$

$$u = 0.15\ c + 0.90\ c = 1.05\ c.$$

where c is the speed of light. Since u is greater than the speed of light, either the Galilean velocity addition is wrong or Maxwell's electromagnetic theory is wrong. Maxwell's electromagnetic theory predicts that the speed of light is independent and has a definite value $c \approx 3 \times 10^8$m/s. However, Maxwell's theory, which was proven correct in numerous experiments conducted under all sorts of conditions, would be contradicted because then light would propagate at different speeds in different reference frames that moved with respect to the ether, which was believed at the end of the nineteenth century to be the hypothetical medium in which light moved. The Michelson-Morley experiment in Section 2.1.4 below shows the absence of the ether as well as the invariance of c. Besides, as we will see in Section 2.1.1 below, the Galilean velocity addition violates Einstein's second postulate of the special theory of relativity. Before introducing Einstein's special theory, which will solve this problem and revolutionize our understanding of space and time, we examine how the Galilean transformation expresses the relationship between an event as measured in the S frame and the same event as measured in the S' frame.

1.2.2 The Galilean Transformation

The Galilean transformation is a set of relations that allows the measurements of an event obtained in one inertial frame S with coordinates $\{x, y, z, t\}$ to be converted into measurements of the same event in another inertial frame S' defined by $\{x', y', z', t'\}$. These measurements are called **the Galilean transformation**.

$$\boldsymbol{x = x' + vt'} \qquad \boldsymbol{x' = x - vt} \qquad \textbf{(2)}$$

$$y = y' \qquad y' = y$$

$$z = z' \qquad z' = z$$

$$t = t' \qquad t' = t$$

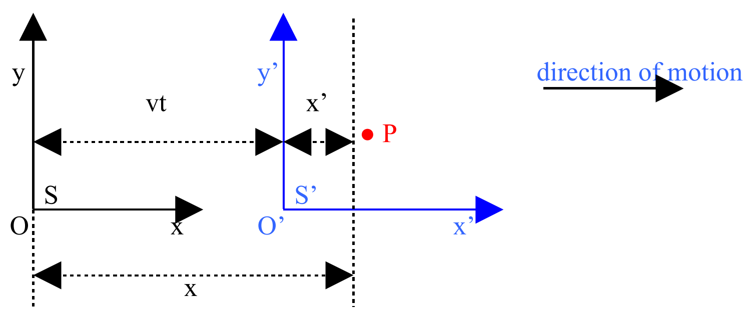


Figure 1-2. Typical relativity situation: stationary frame S, frame S' moving at v relative to S, event P in S'. After time t, P has moved a total of $x = vt + x'$ with respect to S.

Let us see how this transformation came about.

Figure 1-2 shows a typical situation, a frame at rest *S*, a moving frame *S'*, and an event (object, particle, spacecraft) *P* moving in *S'*. We assume that inertial reference frames *S* and *S'* have their *x*-axes along the same line. Let the origin *O'* of frame *S'* move with constant velocity *v* in the direction of the common *x*-axis, i.e., to the right. Suppose for simplicity that at time $t = t' = 0$ the two points of origin coincide, so that at some later time *t'*, they are separated by the distance measured by the product of the velocity *v* and the time elapsed *t'* or *vt'*. An event denoted *P* moving with frame *S'* could then be described by either the coordinates of frame *S*, $\{x, y, z\}$, or those of frame *S'*, $\{x', y', z'\}$. Its *x* coordinate in the *S* frame is then:

$$x = x' + vt'$$

and its *x'* coordinate in the *S'* frame is simply the inverse:

$$x' = x - vt$$

The minus sign makes sense because relative to an event *P* in frame *S'*, which moves rightward, frame *S* moves to the left. Given that only the distance along the *x*-axis changes, and there is no change along the *y*-axis or the *z*-axis, *P*'s *y* and *z* values are unchanged in both frames:

$$y = y'$$

$$z = z'$$

And since in classical physics both time and space are considered absolute, clocks in the two inertial frames show the same time. Hence:

$$t = t'$$

Note that in the first set of relations, all the values on the left-hand side are unprimed and the values on the right-hand side are primed. Just the reverse is true in the second set of relations. The equations in the second set are directly derivable from the first set. These correspondences show the underlying assumptions of classical mechanics, namely that both time and space, as said above, have universal and absolute values. For example, the equation $t = t'$ implies that time is the same regardless of reference frames. As we saw earlier, these transformation equations hold true only at velocities much less than the speed of light, $v << c$. However, at speeds approaching the speed of light, they fail, as we have seen in Section 1.2.1 above.

To see how the equations of the Galilean transformation (2) are related to the Galilean velocity addition (1), consider the point *P* moving in the *x*-direction with constant velocity *v* relative to frame *S* again. Its velocity *u* relative to frame *S* is $u = \Delta x / \Delta t$, and its velocity *u'* relative to frame *S'* is $u' = \Delta x' / \Delta t'$. Intuitively, these are related to equation (1), $u = u' + v'$, of the Galilean velocity addition.

We could derive the equation (1) from the equation (2). Suppose the particle *P* is at coordinate x_1 or x_1' at time t_1, and at coordinate x_2 or x_2' at time t_2. The elapsed time is then $\Delta t = t_2 - t_1$. From Equation (2), we get:

$$\Delta x = x_2 - x_1 = (x_2' - x_1') + v(t_2 - t_1) = \Delta x' + v\Delta t$$

$$\Delta x / \Delta t = \Delta x' / \Delta t' + v$$

$$u = u' + v \qquad (1)$$

As seen above, this last equation of velocity addition poses a serious problem. It allows the speed of light to be exceeded, contrary to the special theory of relativity's second postulate. The Galilean velocity addition allows *u* to be greater than *c*, if *u'* is very close to *c*. However, for everyday events where speeds are much less than *c*, the Galilean-Newtonian framework works perfectly well.

Now that we have discussed the Galilean principle of relativity and its shortcomings at speeds close to the speed of light, we will turn to Einstein's theory of relativity and see how it revolutionized our view of space and time.

As a review, let us apply the equations above to a few examples.

Example 1. A passenger on a train moving at 100 km/h is walking toward the front at a speed of 5 km/h. What is the speed of the passenger with respect to an observer standing on the ground as the train passes?

Solution. This is a classical velocity addition. The train is the moving frame *S'* whose speed is v = 100 km/h carrying a passenger whose speed is u' = 5 km/h, and the observer is at rest in frame *S*. Applying the Galilean velocity addition formula yields the passenger's speed relative to the stationary ground observer:

$$u = u' + v$$

$$u = 5 \text{ km/h} + 100 \text{ km/h} = \underline{105 \text{ km/h.}}$$

Example 2 A train traveling at 120 km/h carrying a passenger who is walking toward its rear at 4 km/h. To a stationary observer on the ground, how fast is the passenger moving?

Solution. This example is analogous to Example 1. However, instead of moving in the direction of the train, the passenger moves in the opposite direction making his speed a negative value. The same velocity addition applies:

$$u = u' + v$$

$$u = -\ 4 \text{ km/h} + 120 \text{ km/h} = \underline{116 \text{ km/h.}}$$

Example 3 A ground observer at rest measures the speed of an airline passenger in flight walking toward the front of the airplane to be 1026 km/h. Knowing the airplane's speed to be 1020 km/h, how fast is the passenger moving?

Solution. This example is analogous to the other examples above. Applying the velocity addition equation, we get:

$$u = u' + v$$

$$1026 \text{ km/h} = u' + 1020 \text{ km/h}$$

$$u' = 1026 \text{ km/h} - 1020 \text{ km/}h = \underline{6 \text{ km/h.}}$$

Example 4 Consider the frame *S* and *S'* at $t = t' = 0$, when *O* and *O'* coincide (Refer to Figure 1-2). Frame *S'* moves with the speed v = 45 m/s with respect to the frame *S*. A particle *P* in *S'* finally comes to rest at coordinates x' = 30 m, y' = 25m, and z' = 0. Calculate the position of *P* with respect to *S* (x, y, z) at (a) t = 2.0 s, and (b) t = 90.0 s.

Solution. (a) We use the Galilean transformation equation (2) to get the x coordinate at t = 2.0 s:

$$x' = x - vt \qquad (2)$$

$$y' = y$$

$$z' = z$$

$$30 \text{ m} = x - 45 \text{ m/s x } 2 \text{ s} = x - 90 \text{ m}$$

$$x = 30 \text{ m} + 90 \text{ m} = \underline{120\text{m.}}$$

Since the motion of *P* is only in the x-direction, its y and z coordinates remain unchanged: y' = 25m, and z' = 0.

(b) Applying the Galilean transformation again, we get the x coordinate t = 90.0 s:

$$30 \text{ m} = x - 45 \text{ m/s} \times 90 \text{ s} = x - 405 \text{ m}$$

$$x = 30 \text{ m} + 405 \text{ m} = \underline{435\ m}.$$

Again, *P*'s *y* and *z* coordinates remain unchanged: $y' = 25$m and $z' = 0$.

2. Special Relativity

2.1 Einstein's Postulates

In a paper of 1905, Einstein introduced an elegant and simple theory to show that time and space are not absolute, but vary depending on the reference frames. He called it **the special theory of relativity** because it applies only to a special set of reference frames called **inertial, i.e., frames in which everything is at rest or moving at a constant, non-accelerated velocity**. We will see in Chapter 3 that the special theory is only a special case of the general theory, which deals with all sorts of motions, including accelerated motions.

The special theory of relativity is based on two simple postulates. Taken separately, the two postulates that form the foundation of the special theory seem innocuous. The first postulate is a generalization of the classical relativity principle, merely extending its application to all of physics, including mechanics, electrodynamics, optics, and thermodynamics. Even the second postulate positing the speed of light as constant regardless of the motion of any frames of reference, though requiring adjustment of our view of the physical world, does not tax our credulity excessively. And although the mathematics involved change accordingly, they are remarkably simple. It is only when applied together that the postulates produce startling results that revolutionize our understanding of space, time, and the universe.

We will see that instead of being absolute and unchanging, time changes with reference frames. A minute in your living room does not have the same "duration" as a minute in a jet plane flying at Mach 1. It is because a traveling clock runs slower than a stationary one, a phenomenon called **time dilation**. Not only do clocks run slower in fast-moving frames of reference, all physical processes also run slower. A person's heartbeat, metabolism, aging process slow down accordingly. Is this an illusion, or is it the nature of time? It is not an illusion. One consequence of time dilation is that given appropriately advanced technology astronauts in the distant future could travel through intergalactic space within their lifetime and come back to earth younger than the friends they had left behind. This certainly is a mind-boggling and tantalizing prospect for mankind and fertile ground for science-fiction imaginings in the meantime.

Another consequence is that there is no such thing as absolute simultaneous events. Events that are perceived as simultaneous in one reference frame are not simultaneous in another. But first, what are simultaneous events? Einstein gave this definition: "If I say, for example, "the train arrives here at 7," this means the coincidence of the small hand of my watch with the number 7 and the arrival of the train are simultaneous events." (Calle, 2002, p. 460) Then he clarified that such a definition is not satisfactory when events occurring at widely dispersed places are considered. Let us take a trivial case. How can we say that an event occurring in Tokyo and another event occurring in New York are simultaneous? Suppose a person in Tokyo and her friend in New York wanted to light up their Christmas trees at the same time and communicated via telephone to synchronize their actions. They agreed that at the count of three both would flip the switch on instantly. And they did. Can we say that these two events were simultaneous? No, because the telephone conversation had to take time to travel across thousands of miles, and by the time friend B heard the count of three from friend A, the latter's clock had shown a later time and she had already thrown the switch, even if their clocks had been synchronized. Hence the events cannot be described as simultaneous. However, this example is a rather weak one for both friends are still considered to be within the same inertial reference frame of the earth. In our thought, it is possible to visualize them synchronizing their clocks at a distance and experiencing simultaneous events, especially when higher speeds are involved. The situation is quite different when they are in different reference frames, e.g., one on Earth and the other traveling in a spaceship. We will see that in these cases **simultaneity is relative**.

Yet another consequence of special relativity appears in the contraction of distance, called **length contraction**. A moving meter stick is shorter than a meter stick at rest. The 20-cm-diameter plate on which your friend's dinner is served on board a rocket ship flying past your space station shrinks relatively to you. In this respect, time and space seem to be a continuum so that it makes sense to talk about spacetime as the four dimensions of the universe, the three spatial dimensions and the fourth dimension, which is time. Time dilation and length contraction are just manifestations of the same consequence of special relativity: you gain in time what you lose in space when seen from different reference frames. Yet, in each frame of reference, all laws of physics hold true because all inertial reference frames are equivalent. For most of us, time is so different from space that calling it the fourth dimension goes against common sense or intuition. We do not even measure time in the same way we measure length. Yet time and space impinge on our everyday life in inextricable ways. Suppose your friend and you have a date to meet on the 10^{th} floor of the Empire State Building in New York City at the corner of 34^{th} Street and Fifth Avenue at 3 pm Saturday, September 24,

2005. The event is defined by the three dimensions of space, one vertical and two horizontal, with time making up the fourth. This event cannot possibly take place except in these four dimensions. This is why we represent it as a system of four coordinates $\{z, y, z, t\}$ for mathematical discussion. There is no complete description of motion without time. In order to gain a complete description, we must specify how a body changes its position over time. To convince yourself, just walk from your bedroom to the kitchen and see if the clock shows the same time at the beginning as at the end of your trip. If the clock shows the same time, you don't need t to describe your motion. But the clock does show a different time, and you are moving both in space and in time.

Now imagine yourself sitting comfortably in an armchair in your living room reading a book from 8 to 9 one evening. To you nothing in your living room moves. Your position is defined spatially by $\{z, y, z\}$ and you may think time has nothing at all to do with your position. In reality you are traveling not only through space but also through time. The event of your reading takes place in the same space in which the earth is waltzing around the sun for one hour. You may be "at rest" with respect to the reference frame of the earth, but the earth is moving from the reference frame of an observer on an asteroid flying by or from the sun's point of view. Your position "at rest" actually alters with time because the earth moves at $v = 3.0 \times 10^4$ m/s. You are in a different location from where you were a second ago. There is nothing you can do to extricate time from the spatial dimensions.

However, in our everyday experience, when speeds are much smaller than the speed of light, $v << c$, the effects of time dilation and length contraction are too small to be perceptible, in the order of nanoseconds and nanometers. Even our fastest rockets seem to crawl when compared to light. Someday when human civilization is capable of designing machines that can accelerate continuously to very high speeds, e.g., speeds approaching light velocity, all of the predictions of the special theory will be commonplace, just as the classical laws of inertia and of universal gravitation are commonplace today. But before we got carried away by the thought of space travel at high velocity, we must realize that in order to accelerate to near c we must produce infinite energy.

2.1.1 Einstein's Two Postulates of Special Relativity

Building on the achievements of Galileo Galilei and Newton, Einstein grounds his special theory on two simple postulates. With these principles the conflict between Newtonian mechanics and Maxwell's electromagnetic theory is elegantly resolved.

The *first postulate*, **the principle of relativity**, states:

The laws of physics are the same in all inertial frames of reference.

This postulate is easy to accept. After all it conforms to our everyday experiences and expectations. For instance, we would expect the laws of physics (including mechanics, electromagnetism, thermodynamics, optics) to work similarly in an environment at rest as well as in one moving uniformly. We would not expect to see coffee flying out of our cup when we fly in an airplane (assuming the plane is not going through a weather disturbance or accelerating) any more than we would expect to see it do the same in our kitchen. Also it is impossible to distinguish one inertial reference frame from any other because the laws of physics apply equally well in any of them. Thus no inertial reference frame, whether at rest or in motion, is preferred over another. The notion of absolute motion and absolute rest loses all significance. We conclude that **motion is relative**.

However, it is the *second postulate*, called **the principle of the constancy of the speed of light**, that is harder to accept.

Light propagates through empty space with a definite speed *c* independent of the motion of the source or of the observer.

According to Maxwell's electromagnetic equations, the speed of light c derives from the formula:

$$c = 1 / \sqrt{\varepsilon_0 \mu_0}$$

where the constant ε_0 is the **permittivity of vacuum** (or free space) and the constant μ_0 is the **permeability of vacuum**, both being used in electric calculations. What is remarkable is that this formula uses electrical methods to calculate the speed of light instead of a direct measurement.

$$c = (\varepsilon_0 \mu_0)^{-1/2} = \left[\left(8.8542 \times 10^{-12} \frac{C^2}{N \cdot m^2} \right) \left(4\pi \times 10^{-7} \frac{N \cdot s^2}{C^2} \right) \right]^{-1/2}$$
$$= 2.9979 \times 10^8 \text{ m/s.}$$

The astounding feat achieved by using electromagnetic methods to establish the speed of light so precisely leads to the acceptance of light as an electromagnetic wave. This success ranks Maxwell's electromagnetic theory as a great theoretical achievement in physics alongside Newton's laws of motion and Einstein's theory of relativity. In 1982, using a highly stabilized laser, scientists derived the speed of light from the formula velocity = wavelength x frequency, $v = \lambda f$ as:

$$\mathbf{c = 299{,}792{,}458 \pm 1.2 \text{ m/s}}.$$

For most purposes, we use $c = 3.00 \times 10^8$ m/s. As scientists are constantly tinkering with light, they are able to slow it down (Shirber, 2004), stop it, or push it beyond c. But this is another story.

In 1983 the Seventeenth General Conference on Weights and Measures adopted a definition of the meter based on the above speed of light:

The meter is the length of path traveled by light in vacuum during a time interval of 1/299,792,458 of a second.

Despite all the convincing calculations of the speed of light, the second postulate still runs counter to our common sense notions. That the speed of light is independent of the motion of its source is not problematic. In this respect light behaves just as sound does. Once emitted, light and sound become independent of their source or its motion. Sound speed then depends on the medium in which it propagates. And before the twentieth century, scientists believed that light would be similarly affected by the hypothetical ether wind, called the **luminiferous ether**, in which it traveled. We will see how this hypothesis was shattered by the Michelson-Morley experiment in Section 2.1.4 below.

Yet, we find it hard to believe that a person traveling toward or away from a light source, or just staying motionless measures the speed of light in vacuum always to be $c = 3.00 \times 10^8$ m/s. Our intuition would require us to add or subtract the speed of the observer depending on whether the observer is moving in the same direction as the light source or in the opposite direction. Now imagine two spaceships: Ship 1 is at rest with respect to an outside source of light, Ship 2 is moving with velocity v toward the same outside beacon. Both ships measure the light from the beacon as well as the light from their own internal sources. Ship 1 measures the speed of internal light as well as that of the beacon as c, as is expected from the first postulate. What about Ship 2? It measures the speed of the internal light as c (according to the second postulate), but it must measure the beacon's light as having speed c also, not $c - v$; for otherwise, both sources of light would be used as a motion detector to determine that Ship 2 is moving, in violation of the first postulate. Recall that according to the first postulate all inertial reference frames are equivalent, i.e., no reference frame is preferred over any other reference frame.

We conclude that **regardless of the speed of an inertial observer's motion toward or away from the source of light, and regardless of the speed of a light source's motion toward or away from the inertial observer, the speed of light in vacuum is always the constant c**.

Example 5. Astronaut A piloting a spaceship traveling eastward with velocity 0.80 c sends a light beam forward. Some distance away Astronaut B in another spaceship travels westward toward the first spaceship with velocity 0.85 c. With what speed does Astronaut B see the light beam from Astronaut A pass by?

Solution. The speed of light in free space is always c no matter how fast or slow an inertial observer (Astronaut B) moves toward or away from it and no matter how fast or slow its source (Astronaut A) moves toward or away from the inertial observer. This result is also shown mathematically in Example 8 below.

The two postulates at the heart of Einstein's special theory of relativity, though simple in each formulation, revolutionize our concept of time and space. By giving up the basic assumption of Newton's laws that space and time are absolute and independent of anything external to them, the special theory reconciles Maxwell's electromagnetism and Newtonian mechanics in a simple way.

In the remaining sections of this chapter, we examine the arguments and experiments that corroborate the special theory, and discuss its amazing consequences. But first, since we have seen that the Galilean transformation works only for speeds much less than the speed of light, how do we convert measurements from one inertial system to measurements in another, one or both of which move at speed close to the speed of light? This leads to a set of equations, called the Lorentz transformation, which Einstein derived independently.

2.1.2 The Lorentz Transformation

Within the framework of classical physics, where speed is much smaller than the speed of light, $v << c$, the Galilean velocity addition and transformation equations are perfectly adequate. However, they will not work with speeds close to c.

For high speed, the Galilean transformation will be replaced by a new set of transformation equations called **the Lorentz transformation**. If we examine the familiar two inertial frame systems, S and S', we can assume that the new equations will also be linear. Let us assume that the new equations differ from the Galilean equations by a constant factor γ and have the forms:

$$x = \gamma(x' + vt'), \qquad y = y', \qquad z = z'$$

The constant factor γ is to be determined; y and z do not change, and t will be derived. Of course, the converse set will be:

$$x' = \gamma(x - vt), \qquad y' = y, \qquad z' = z$$

Consider a light beam that leaves the common origin of S and S' at time $t = t' = 0$. After a time t it will have traveled a distance $x = ct$ *or* $x' = ct'$ along the x-axis. Substituting these values in the first equations above, we get:

$$ct = \gamma(ct' + vt') = \gamma(c + v)t'$$

$$ct' = \gamma(ct - vt) = \gamma(c - v)t$$

$$t' = \gamma(c - v)t / c$$

By substituting t' in the first equation, we obtain:

$$ct = \gamma(c + v)t' = \gamma\,(c + v)\,\gamma\,(c - v)\;t / c$$

$$ct = \gamma^2\,(c^2 - v^2)\,t / c$$

from which we derive γ, after canceling t:

$$\gamma^2 = c / (c^2 - v^2) / c$$

$$\gamma^2 = c^2 / (c^2 - v^2)$$

$$\gamma^2 = 1 / (1 - v^2 / c^2)$$

$$\boldsymbol{\gamma = 1 / \sqrt{1 - v^2 / c^2}} \qquad \textbf{(3)}$$

Using γ, we now find the relation between t and t'. We substitute x in x':

$$x' = \gamma(x - vt) = \gamma\,[\gamma\,(x' + vt') - vt] = \gamma^2\,(x' + vt') - \gamma\,vt$$

From known equations above,

$$ct = \gamma(ct' + vt')$$

and

$$x' = ct' \text{ or } t' = x' / c$$

we derive:

$$ct = \gamma(ct' + vx' / c)$$

$$t = \gamma\,(t' + vx' / c^2)$$

$$t = 1 / \sqrt{1 - (v^2 / c^2)}\ (t' + vx' / c^2)$$

Similarly, we derive *t'* and obtain:

$$t' = 1 / \sqrt{1 - (v^2 / c^2)}\ (t - vx / c^2)$$

The **Lorentz transformation equations** are then:

$$\boldsymbol{x = \gamma\,(x' + vt')} \qquad \boldsymbol{x' = \gamma\,(x - vt)} \qquad \textbf{(4)}$$

$$y = y' \qquad y' = y$$

$$z = z' \qquad z' = z$$

$$\boldsymbol{t = \gamma\,(t' + vx' / c^2)} \qquad \boldsymbol{t' = \gamma\,(t - vx / c^2)}$$

These equations are slightly different from the original transformation proposed by Lorentz to account for the null result of the Michelson-Morley experiment (Section 2.1.4 below). Einstein (1961, pp. 115-122) arrived at his Lorentz transformation equations by a totally different derivation. *The Lorentz transformation equations are nothing but the relativistic generalization of the Galilean transformation equations.*

As can be surmised, **the factor γ plays a crucial role in relativistic measurements**.

$$\boldsymbol{\gamma = 1 / \sqrt{1 - (v^2 / c^2)}} \quad \textbf{in s} \qquad \textbf{(3)}$$

or, since $\beta = v / c$,

$$\boldsymbol{\gamma = 1 / \sqrt{1 - \beta^2}} \quad \textbf{in s} \qquad \textbf{(3)}$$

The spreadsheet formula for γ is:

γ **= (1-(v/c)^2)^-(1/2)** **(5)**

Substitute actual values for *v* and *c* in the formula (5) to get the result. If *v* is expressed in terms of *c*, e.g., 0.95 *c*, then this *v* value takes the place of v/c in the formula.

And **the spreadsheet formula for the reciprocal 1/ γ or γ^{-1} of γ** is:

γ^{-1} **= (1- (v/c)^2)^(1/2)** **(6)**

Remember that an Excel formula begins with an equal sign. Just substitute the real values for *v* and *c*. Table 2-3 below is constructed using this spreadsheet formula.

To simplify the calculations, always try to reduce v to a fraction of c. For example, if $v = 3 \times 10^7$ m/s, make it $v = 0.10\ c$. This insures that c^2 will not enter into the calculations at all. And the spreadsheet formulas for γ and γ^{-1} reduce to:

$$\gamma = (1\text{-}v\hat{}2)\hat{}\text{-}(1/2) \qquad \textbf{(5a)}$$

and

$$\gamma^{-1} = (1\text{- }v\hat{}2)\hat{}(1/2) \qquad \textbf{(6a)}$$

respectively.

γ in Relativistic Measurements

Since γ is the factor that affects all relativistic measurements such as relativistic length, time, energy, mass, and momentum, let us examine γ in some detail. Its reciprocal γ^{-1} is easily derived. Just bear in mind that $\gamma >= 1$ and $\gamma^{-1} <= 1$.

(1) In γ's formulation, v cannot be greater than c, for otherwise the value under the radical sign would be negative resulting in an imaginary number. However, in the 1960's, some scientists pointed out that there is nothing in γ, wherever it may be used, that prevents v being greater than c. They hypothesized the existence of a fast particle called "tachyon" (meaning "fast"), whose speed $v > c$. If very fast particles existed, their rest mass m_0 would have to be an imaginary number, in the mass formula $m = m_0\gamma$, in order for their mass m to be real. So far the hypothetical tachyons have not been found. Thus, the speed of light is still the ultimate speed of the universe.

(2) When $v = c$, the value under the radical is equal to 0, and γ is not defined. Therefore, v cannot equal c.

(3) If $v = 0$, or $v << c$, $\gamma = 1$. All objects and events are at rest. Therefore, there are no relativistic effects.

(4) The ratio v / c, also called β, controls the value of γ. If v goes to infinity (an impossibility by the second postulate), γ is an imaginary number as the value under the radical is negative. We saw this in (1) above. If c becomes infinity (which it can never be), $\gamma = 1$, and there are no relativistic effects.

(5) When solving relativistic problems, convert v into a fraction of c to simplify calculations. Thus, β reduces to v in terms of c.

Since the ratio v / c determines the value of γ, let us examine a few examples of this ratio, γ and its reciprocal, all of which occur in relativistic measurements. (Hecht, 1998, p. 964):

Relations between β, $1/\gamma$, and γ

$\beta = v / c$	$1/\gamma = \sqrt{1-(v^2/c^2)}$	$\gamma = 1/\sqrt{1-(v^2/c^2)}$
0.000 000	1.000 000	1.000 000
0.100 000	0.994 987	1.005 038
0.200 000	0.979 796	1.020 621
0.300 000	0.953 939	1.048 285
0.400 000	0.916 515	1.091 089
0.500 000	0.866 025	1.154 701
0.600 000	0.800 000	1.250 000
0.700 000	0.714 143	1.400 280
0.800 000	0.600 000	1.666 667
0.900 000	0.435 890	2.294 157
0.990 000	0.141 057	7.088 812
0.999 000	0.044 710	22.366 27
0.999 900	0.014 142	70.712 45
0.999 990	0.004 472	223.607
0.999 999	0.001 414	707.107
1.000 000	0.000 000	division by zero

Table 2-1. Relations between *β*, 1 / *γ*, and *γ*. These important relations recur wherever relativity measurements are made, be they relativistic length, time, mass, energy, or momentum.

Since $\beta = v / c$, the ratio of the speed of a moving object to the speed of light, should never reach 1, γ enforces, so to speak, the upper limit for all speeds, which can never be reached except for light. The above table also shows that at everyday speed, which is very small compared to c, $v << c$, the Lorentz transformation reduces to the Galilean transformation because γ has no effect, being very close to 1. In other words, the Galilean relativity principle is a special case of the first postulate of the special theory of relativity.

Given that the values β and γ are ubiquitous in relativity problems, let us remember that $\boldsymbol{\gamma > 1}$ and its reciprocal $\boldsymbol{1 / \gamma < 1}$ since v in the ratio $\beta = v / c$ is never zero. To avoid to tediousness of calculating γ when v is very small, i.e., $v << c$, it is useful to work out an approximate expression for γ. We use a binomial formula to this end.

A **binomial formula** has this form:

$$(a + b)^n = a^n + \frac{n}{1} a^{n-1}b + \frac{n(n-1)}{1\cdot 2} a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} a^{n-3}b^3 + \ldots +$$

$$\frac{n(n-1)(n-2)\ldots(n-r+1)}{1\cdot 2\cdot 3\cdot\cdot\cdot\cdot\cdot r} a^{n-r}b^r + \ldots + b^n$$

Since the formula for $\gamma = (1 - \beta^2)^{-1/2}$ is a binomial, we apply the binomial expansion above making $a = 1$, $b = -\beta^2$ and $n = -½$, retaining only the first two terms, and treating the rest of the terms as negligible:

$$(1 + b)^n \approx 1 + n(a^{n-1}b) \approx 1 + n(1^{-1/2-1})(b) \approx 1 + nb$$

Hence:

$$\gamma \approx 1 + nb$$

Substituting the values for n and b in the above relation yields **the shortened binomial formula of** $\boldsymbol{\gamma}$:

$$\gamma \approx (1 - \beta^2)\ 1 + (-1/2)(-\beta^2)$$

$$\boldsymbol{\gamma \approx 1 + \beta^2 / 2} \qquad \textbf{(7)}$$

The spreadsheet formula of relation (7) is:

$$\boldsymbol{\gamma = 1+((\beta\hat{}2)/2)} \qquad \textbf{(8)}$$

Substitute an actual value for β in the formula (8) to get the result. Since $\beta = v/c$, if v is expressed in terms of c, e.g., 0.95 c, then this v value takes the place of β in the formula.

This approximate value of γ comes in handy when we need to quickly check our calculation results when $v << c$, e.g., up to around $v <= 0.6\ c$.

For v closer to c, $v \approx c$ or $\beta \approx 1$, the following **approximation** should be used:

$$(1 - \beta^2) = (1 - \beta)(1 + \beta) \approx 2(1 - \beta)$$

$$\boldsymbol{\gamma \approx 2(1 - \beta)^{-1/2}} \qquad \textbf{(9)}$$

The spreadsheet formula of relation (9) is:

$$\boldsymbol{\gamma = 2*(1\text{-}\beta)\hat{}(\text{-}1/2)} \qquad \textbf{(10)}$$

Substitute an actual value for β in the formula 10 to get the result. Since $\beta = v/c$, if v is expressed in terms of c, e.g., 0.95 c, then this v value takes the place of β in the formula. In other words, **if v is expressed as a fraction of c, then $\beta = v$**.

Given the importance of γ and its component β, let us list a few known speeds and their corresponding γ and β values in Table 2-2. (Slightly modified from Hecht, 1998, p. 958)

Speeds of objects and their corresponding γ and β values

Object	**Speed v**	**$\beta = v / c$** (v in terms of c)	**$\gamma = (1 - (v^2 / c^2))^{-1/2}$**
Human walking	8 km/h	0.000 000 007	1 000 000 000
100-yard dash (max.)	10.0 m/s	0.000 000 033	1.000 000 000
Commercial automobile (max.)	62 m/s	0.000 000 21	1 000 000 000
Sound	333 m/s	0.000 001 11	1 000 000 000
SR-71 reconnaissance jet	980 m/s	0.000 003 27	1.000 000 000
Moon around Earth	1000 m/s	0.000 003 33	1.000 000 000
Space shuttle	7599 m/s	0.000 025 33	1.000 000 000
Apollo 10 (re-entry)	11.1 km/s	0.000 037	1.000 000 001
Escape speed (Earth)	11.2 km/s	0.000 037	1.000 000 001
Pioneer 10	14.4 km/s	0.000 048	1.000 000 001
Earth around Sun	29.6 km/s	0.000 099	1.000 000 005
Mercury orbital speed	47.9 km/s	0.000 16	1.000 000 013
Helios B solar probe	66.7 km/s	0.000 22	1.000 000 025
Earth-Sun around galaxy	2.1×10^5 m/s	0.000 70	1.000 000 245
Electrons in a TV tube	9×10^7 m/s	0.3	1.05
Muons at CERN	2.996×10^8 m/s	0.999 4	28.87
Electrons at Stanford Linear Accelerator (SLAC)	$2.997\,9 \times 10^8$ m/s	0.999 991 800 995	246.95

Table 2-2. Speeds of objects and their corresponding γ and β values

Now let us construct a more comprehensive table with up to 15 decimal places, with v as close to c as the Excel spreadsheet allows. It starts with the speed near that of the space shuttle, the highest speed achieved by a manned flying machine. Even with the speed of a space shuttle of 17,000 mi/h, 27,358 km/h, or 7.599 km/s (bolded in the following table), $\gamma \approx 1$. This high speed is only about 1/39,541th the speed of light. As the speed v of an object gets very close to c, its γ value increases dramatically (See Table 2-3 below).

Caution: *The binomial expansion of γ is, for practical purposes, fairly reliable only for $v << c$, e.g., for $\beta = v / c \approx 0.60$ because of the retention of only the first two terms of the expansion. Beyond this β value, the second approximation formula 9 gives results that are remarkably similar to those obtained with the full γ formula, as the table below (Table 2-3) clearly shows. When in doubt, use the full formula or its spreadsheet equivalent. However, if you have a calculator and want only a quick, reasonably acceptable result, use the binomial formula for $v <= 0.60\ c$ and the second approximation for $v > 0.60\ c$. Just bear in mind that even the calculator will tackle the full γ formula without undue trouble.*

For convenience, a spreadsheet version of the full γ expression appears in Formula (5), and a spreadsheet version of its reciprocal γ^{-1} is Formula (6) above.

Table 2-3 below lists a number of $\beta = v / c$ values with up to 15 decimal places and their corresponding γ values derived by the shortened binomial formula (Formula 7, rows 1-12), the second approximation (Formula 9, row 13 to end), and the full γ formula (Formula 3 in the third column). Note that from row 24 to the end of the table, the second approximation and the full γ formula yield identical results. This table represents to best application of the binomial and approximation formulas.

Discrepancies between the shortened binomial expansion of γ and the γ formula

$\beta = v / c$	**$\gamma \approx 1 + (v^2 / c^2) / 2$**	**$\gamma = (1 - (v^2 / c^2))^{-1/2}$**

(*v* in terms of *c*)	(binomial expansion, for *v*<< *c*) $\gamma \approx \mathbf{2(1-\beta)^{-1/2}}$ (for *v* ≈ *c*, rows 13 and following)	(in seconds)
0.0000200000000000	1.0000000002000000	1.0000000002000000
0.000025330000000	***1.0000000003208000***	***1.0000000003208000***
0.0002000000000000	1.0000000200000000	1.0000000200000000
0.0020000000000000	1.0000020000000000	1.0000020000060000
0.0200000000000000	1.0002000000000000	1.0002000600200100
0.1000000000000000	1.0050000000000000	1.0050378152592100
0.2000000000000000	1.0200000000000000	1.0206207261596600
0.3000000000000000	1.0450000000000000	1.0482848367219200
0.3700000000000000	1.0684500000000000	1.0763894728677100
0.4000000000000000	1.0800000000000000	1.0910894511799600
0.5000000000000000	1.1250000000000000	1.1547005383792500
0.6000000000000000	1.1800000000000000	1.2500000000000000
0.7000000000000000	**1.2909944487358100**	**1.4002800840280100**
0.8000000000000000	1.5811388300841900	1.6666666666666700
0.9000000000000000	2.2360679774997900	2.2941573387056200
0.9900000000000000	7.0710678118654700	7.0888120500833500
0.9990000000000000	22.3606797749979000	22.3662720421294000
0.9994000000000000	28.8675134594802000	28.8718445610217000
0.9999000000000000	70.7106781186586000	70.7124459519145000
0.9999900000000000	223.6067977504880000	223.6073567696250000
0.9999990000000000	707.1067811763810000	707.1069579492320000
0.9999999000000000	2 236.0679780882700000	2,236.0680339452900000
0.9999999900000000	7 071.0677941002700000	7,071.0678137264200000
0.9999999990000000	**22,360.6800911995000000**	**22,360.6800911995000000**
0.9999999997000000	40,824.8273574557000000	40,824.8273574557000000
0.9999999999000000	70,710.6751933411000000	70,710.6751933411000000
0.9999999999900000	223,606.7884993250000000	223,606.7884993250000000
0.9999999999990000	707,114.6025254690000000	707,114.6025254690000000
0.9999999999999000	2,235,720.4112652200000000	2,235,720.4112652200000000
0.9999999999999900	7,073,895.3808826000000000	7,073,895.3808826000000000
0.9999999999999990	22,369,621.3333333000000000	22,369,621.3333333000000000

Table 2-3. Shortened binomial formula and the second approximation of γ versus full γ formula for various $\beta = v / c$ values. The shortened binomial formula 7 gives fairly good approximations of γ up to the value of $\beta = 0.60$. From that point on, the second approximation formula 9 takes over and produces remarkably close values of γ up to $\beta = 0.99999999$. Beyond this value, as v further approaches c, the second approximation gives the same values of $\boldsymbol{\gamma}$ as the full formula.

The graph of this table (shown in Figure 2-9, Section 2.3 below) shows how dramatic the rapid rise of γ is as v approaches c. Note that $\beta = v / c$ can never reach c, by the second postulate, i.e., $v \neq c$, so that $\beta >= 0$ and $\beta < 1$, and $\gamma >= 1$and $\gamma^{-1} <= 1$. In addition, the full formula shows that γ would be undefined if $v = c$ because of the division by zero. Note, however, that the truncated binomial formula of γ would not show this effect of $v = c$, which points to another of its undesirable aspects.

2.1.3 Relativistic Addition of Velocities

Using the results obtained in the Lorentz transformation, let us determine **relativistic velocity addition**. As seen above, the classical velocity addition, $u = u' + v$, is valid only at very small speeds compared to c. Consider our familiar pair of inertial reference frames S and S'.

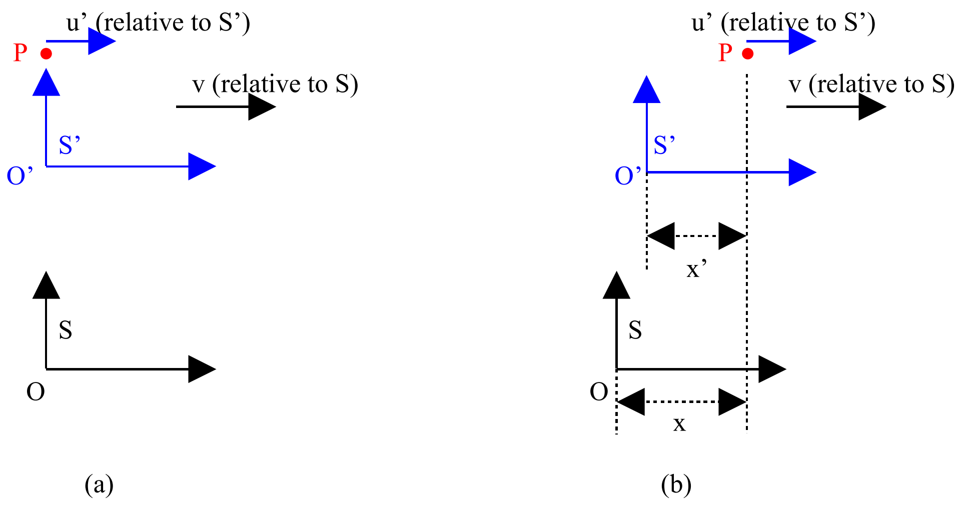


Figure 2-1. Moving event P within moving S' with respect to stationary S.

Suppose at time $t = t' = 0$ the origins O and O' coincide (Figure 2-1a), and the particle P passes the origin at speed $u' = x' / t'$ with respect to frame S' in the $+x$ direction. The S' frame itself moves with constant speed v relative to the S frame. At a later time, the particle is at x in the S frame, where its speed is $u = x / t$ and at x' in the S' frame, where its speed is $u' = x' / t'$ (Figure 2-1b).

Using the Lorentz transformation equation 4, substituting x', canceling γ, and factorizing t', we get:

$$u = \frac{x}{t} = \frac{\dfrac{x' + vt'}{\sqrt{1 - (v^2 / c^2)}}}{\dfrac{t' + (vx' / c^2)}{\sqrt{1 - (v^2 / c^2)}}} = \frac{t'(u' + v)}{t'(1 + u'v / c^2)}$$

By canceling t', we obtain the **relativistic velocity addition equation**:

$$\boldsymbol{u = \frac{u' + v}{1 + u'v / c^2}} \qquad \textbf{(11)}$$

where u is the velocity of a moving object relative to the reference frame of a stationary observer, u' is the velocity of the object with respect to its own reference frame, and v is the velocity of moving frame relative to the stationary frame. The same applies if the observer is also moving along with the object.

The primed counterpart of u has a negative v just like in the Galilean velocity addition:

$$\boldsymbol{u' = \frac{u - v}{1 - uv / c^2}} \qquad \textbf{(12)}$$

Note that the numerator of the relativistic velocity transformation equation is the same as the numerator of the Galilean velocity addition. The difference between these transformations resides in the denominator, which is responsible for making sure that no object can travel at c. And that includes protons, electrons, Starship Enterprise, or any other machine we can devise in the future. When both u and v are equal to c, the denominator equals zero. This is why ***c* is the ultimate speed.**

Let us now explore the applications of the Lorentz transformation equations and the relativistic velocity transformation.

The most important thing to remember when working with relativity problems is to identify the reference frame under consideration. Special relativity deals only with inertial frames, i.e., frames that are at rest or in constant motion. There are three entities involved: (1) the stationary frame S (relative to the other), (2) the moving frame S', and (3) an event, a particle, or an object P moving in S'. In applying the velocity addition formula, keep in mind that u' is the velocity of an object in a frame (S') moving with speed v with respect to the frame (S) in which speed u of P is measured.

Example 6. An astronaut in a spaceship moving away from the earth with a constant speed of 0.9 c fires a rocket in the direction of travel with a speed of 0.5 c relative to the spaceship. What is the speed of the rocket as observed by an earthbound observer?

Solution. Let S be the earth's reference frame, and S' be the reference frame of the spaceship. The spaceship travels at $v = 0.9\,c$, and the rocket at $u' = 0.5\,c$. The question is to determine the speed of the rocket relative to earth, i.e., frame S. As we cannot apply the Galilean velocity addition, since it would violate the second postulate, $0.9\,c + 0.5\,c = 1.4\,c$, we apply the relativistic velocity transformation:

$$u = (u' + v) / [1 + (u'v / c^2)] \qquad (11)$$

$$u = (0.5\,c + 0.9\,c) / [1 + (0.5\,c)\,(0.9\,c) / c^2] = 1.4\,c / [1 + (0.45\,c^2 / c^2)]$$

$$u = 1.4\,c / 1.45 = \underline{0.967\,c.}$$

Example 7. Two stars A and B move away from Earth in opposite directions with velocity $0.9c$ and $0.8c$ respectively. Find the speed of star B with respect to star A.

Solution. First we make star A the S frame (the frame at rest) with respect to which star B's speed is going to be measured. We make Earth the moving frame S' in which Earth is at rest while both stars are moving relative to Earth. However, in frame S star A is at rest, and Earth is moving with velocity $v = 0.9\,c$ relative to star A, and star B is moving at velocity $u' = 0.8\,c$ relative to Earth. We are to find the speed of star B relative to star A.

We apply the relativistic velocity addition equation :

$$u = (u' + v) / [1 + (u'v / c^2)] \qquad (11)$$

$$u = (0.8\,c + 0.9\,c) / (1 + (0.8\,c)(0.9\,c) / c^2)$$

$$u = 1.7\,c / \ 1.72 = \underline{0.988\,c.}$$

Example 8. Show that the relativistic velocity addition equation insures that no moving object can exceed the speed of light c.

Solution. Imagine an astronaut in a spaceship moving at speed v relative to an observer sends a light beam forward. The astronaut measures the light's speed as c. The observer measures the speed of the light beam, and found the result consistent with Einstein's second postulate. Substituting c for u' in the relativistic velocity transformation, we get:

$$u = (u' + v) / [1 + (u'v / c^2)] \qquad (11)$$

and after multiplying both numerator and denominator by c:

$$u = (c + v) / [1 + (cv / c^2)] = c(c + v) / (c + v) = \underline{c}.$$

Example 9. A spaceship travels away from Mars with velocity 0.55 *c* and fires a lander back at Mars at the speed of 0.6*c* relative to the spaceship. What is the speed of the lander as seen from Mars?

Solution. Taking the direction of the spaceship as positive, we apply the relativistic velocity equation. The spaceship moves at $v = 0.550\ c$ and the lander's speed $u' = -\ 0.600\ c$ (negative meaning in the opposite direction). The lander's velocity with respect to Mars is:

$$u = (u' + v) / [1 + (u'v / c^2)] \qquad (11)$$

$$u = (-\ 0.600\ c + 0.550\ c) / [1 + (-\ 0.600\ c)(0.550\ c)\ / c^2]$$

$$u = (-\ 0.050\ c) / [1 - (0.330\ c^2 / c^2)] = -\ 0.050\ c / 0.670 = \underline{-\ 0.074\ c.}$$

Example 10. Rocket ship A and rocket ship B approach their space station from the same direction, ship A with the speed of 0.665 *c* and ship B with the speed of 0.585 *c*. What is the speed of ship A with respect to ship B?

Solution. The first step is to identify the reference frames. Let us choose the space station as the frame at rest *S*. Then both rocket ships are in the moving frame *S'*. Rocket ship A has velocity $u = 0.665c$ and rocket ship B has velocity $v = 0.585\ c$. We apply velocity addition equation (12):

$$u' = (u - v) / [1 - (uv / c^2)] \qquad (12)$$

$$u' = (0.665\ c - 0.585\ c) / [1 - (0.665\ c)\ (0.585\ c)\ / c^2)]$$

$$u' = (0.080\ c) / [1 - (0.389\ c^2\ / c^2)] = 0.080\ c / 0.611 = \underline{0.131\ c}.$$

Example 11. A rocket ship leaves its space station at the speed of 0.4 *c*, and emits a light pulse back toward the space station. At what speed does an observer in the space station see the light pulse coming to her?

Solution. The rocket ship is moving in frame *S'* in the +*x* direction (i.e., away from the space station) at $v = 0.4c$, and the space station is at rest in frame *S*. The navigator aboard the rocket ship measures the light pulse and finds its speed to be $u' = -\ c$, the negative sign designating movement in the –*x* direction (toward the station). Applying the relativistic velocity transformation, we get:

$$u = (u' + v) / [1 + (u'v / c^2)] \qquad (11)$$

$$u = (-\ c + 0.4\ c) / [1 + (-\ c)(0.4\ c) / c^2]$$

$$u = 0.6\ c / (1 - 0.4) = \underline{c.}$$

This is another confirmation of the second postulate. No matter whether the source of light moves away from or toward the observer, the speed of light remains the constant *c*.

Example 12. Galaxy A and Galaxy B move away from Mars in opposite directions at a speed of 0.8 *c* with respect to Mars. Find the speed at which the galaxies more apart relative to each other.

Solution. Again we have the familiar *S* and *S'* frames, and the point *P*, the three objects that require the addition of velocities. Let Galaxy A moving left be the *S* frame, Mars the *S'* frame, then Galaxy B moving right is *P*. A straight application of the relativistic velocity transformation yields:

$$u = (u' + v) / [1 + (u'v / c^2)] \qquad (11)$$

$$u = (0.8\ c + 0.8\ c)\ [1 + (0.8\ c)(\ 0.8\ c\ ) / c^2]$$

$$u = 1.6\ \mathrm{c} / 1.64 = \underline{0.975\ c.}$$

The two galaxies can never separate with a speed equal to or greater than the speed of light. This implies that communication via light, and gravitational and electromagnetic interactions that propagate at *c* are always possible between any two intergalactic bodies because they can never outrun light.

2.1.4 The Michelson-Morley Experiment

At the end of the nineteenth century a great puzzle left behind by Maxwell remained to be solved. Galilean relativity predicted that the speed of light would vary depending on the inertial reference frames. Such was the dominance of Newtonian mechanics that scientists believed that light must travel in some sort of medium just like sound waves and water waves. They postulated the **luminiferous ether**, a hypothetical invisible, massless, thin yet rigid substance that filled space and acted as a medium through which electromagnetic waves must propagate. Strange substance indeed. Yet, even Maxwell himself believed there was such an ethereal medium filling space and permeating everything. The earth too must rotate around its axis and revolve around the sun through the ether. Maxwell's equations hold and light travel at speed *c* only in an absolute reference frame which is at rest with respect to the ether. This is called the **absolute frame.** In other frames of reference the equations gave contradictory results, and needed to be modified to work. This seems to be at odds with the classical relativity principle, whereby no inertial frame is privileged over any other. Consider, for example, that a spaceship traveling at 1.0×10^8 m/s turns on a light beacon, and measures its speed to be 3.0×10^8 m/s. To a stationary observer, the light travels at speed 1.0×10^8 m/s + 3.0×10^8 m/s = 4.0×10^8 m/s. Thus the same electromagnetic effect has different values when measured in different reference frames. But Maxwell's theory does not provide for relative speed. It predicted the speed of light to be $c = 3.0 \times 10^8$ m/s. It seemed intriguing that Maxwell's electromagnetic equations, which worked so successfully and were corroborated by numerous experiments, should have to be an exception to the Galilean relativity principle, which applied to Newtonian mechanics but not to Maxwell's electromagnetic theory. Clearly this dilemma called for investigation to reconcile the classical relativity principle and the electromagnetic theory.

In 1879, the year Einstein was born, James Clerk Maxwell (1831-1879) wrote a letter before he died in which he discussed a potential experiment to measure the speed *v* of the earth as it travels through the ether wind. Many experiments were conducted using the interferometer, but yielded unconvincing results. None was as famous as the one performed by the American physicist Albert A. Michelson. (1852-1931) and the American chemist Edward W. Morley (1838-1923). Born in Strelno, Prussia (now Strzelno, Poland), Michelson at age 4 immigrated with his parents to America. Rejected by the Naval Academy, he managed to gain admission through a "chance" meeting with President Grant, who was impressed with the young man. After graduation, Michelson stayed at the Academy as an instructor of physics and chemistry. In 1878 he was thinking about how to measure the speed of light better, but knew he needed more formal training. So in 1880, he took a leave of absence to pursue advanced studies in France and Germany. While in Europe he came across Maxwell's last letter about measuring the speed of the earth through the ether. Past experimenters had used the interferometer to measure the effects of motion on the transmission of light through the ether. Although they failed to detect the ether, the limited precision of their instruments made their findings less than convincing. Michelson designed a new instrument with a precision that would allow him to settle the question definitively. He hoped to detect the ether wind that carried light waves with his new and improved interferometer. If the ether existed, so went the reasoning, it would act like any wind or stream, impeding or aiding the progress of an object moving in it.

Let us take an example to illustrate the problem. We use the analogy of boats traveling on a river. In Figure 2-2 a boat travels at a fixed speed of c = 5 km/h in the river which flows to the right with a speed of v = 3 km/h.

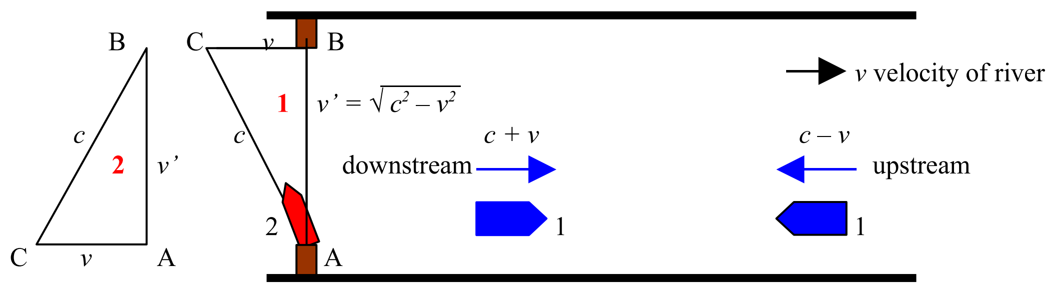


Figure 2-2. The downstream trip made by Boat 1 at velocity $c + v$ is faster than its upstream trip at $c - v$. On the cross-stream trip, Boat 2 starts from A heads for C to land at B (triangle 1). On the return trip Boat 2 starts from B heading for C and is carried back by the stream to A (triangle 2).

Let us compare the travel time of two trips: 4 km downstream and back versus crossing the river and back. A boat (Boat 1) heading downstream travels at velocity c (not the speed of light) augmented by the river's velocity v, moving a total of $c + v$ = 5 km/h + 3 km/h.= 8 km/h. The trip downstream takes: 4 km / 8 km/h = 0.5 h. On the way back, the boat traveling upstream makes slower progress equivalent to $c - v$ or 2 km/h. The trip back, therefore, takes 4 km / 2 km/h = 2 h. The total transit time up and down stream takes: 0.5 h + 2 h = 2.5 h.

Now the boat (Boat 2) crosses the river from pier A to pier B. With the river current flowing to the right, if Boat 2 heads straight across from pier A, it will be dragged downstream and miss pier B. To compensate, Boat 2 has to head into the current (upstream) at a certain angle so that the combined effect of c and v would allow it to reach Pier B. This angle depends on c and v. In Figure 2-2 the width of the river AB, the distance AC traveled by Boat 2, and the distance CB of the effect of the river flow on Boat 2 form the right triangle 1. Likewise, on the return trip Boat 2 starts from Pier B, heads into the stream for C, and reaches A dragged by the downstream current (as depicted by triangle 2). Using the Pythagorean theorem, we can derive the length of any of its side given those of the other two by using the formula $c^2 = v^2 + v'^2$. This geometry shows that by making the trip as depicted Boat 2 *effectively* travels $v' = (c^2 - v^2)^{1/2} = (25 - 9)^{1/2}$ = 4 km, and takes just 1 h (because it travels 5 km at 5 km/h) to make the trip across, and 1 h to come back, a total of 2 h.

Hence, in this case, for the same total distance of 8 km, the boat takes less time crossing the river than going up and down stream. In general, the times for the round trips will differ.

The Michelson-Morley experiment (in fact, many were conducted at various places and times) builds on the boat trip analogy just described. It uses a precision interferometer represented in schematic form in Figure 2-3. This experiment aimed to measure the ether wind with respect to the earth or equivalently the earth's velocity relative to the ether. We will use the analogy of the boat's journey on the river to interpret the result of the experiment. The ether wind is analogous to the velocity of the river flow; and the two boats going up and down stream, and across the river and back are analogous to the light beams 1 and 2 on the perpendicular arms of the interferometer. We would expect that the total travel times of the beams would differ in the same way as the boats' total transit times differed.

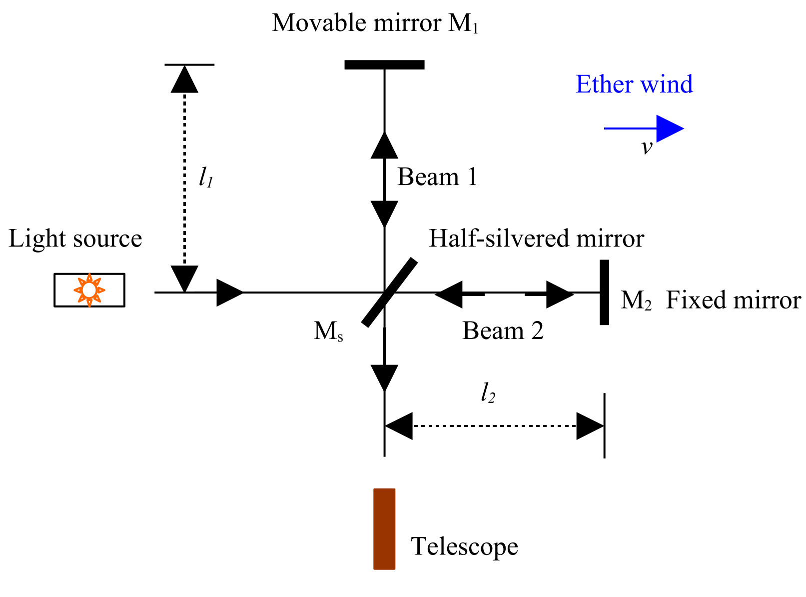


Figure 2-3. The Michelson-Morley experiment. In this interferometer, the light beam travels from its source to the half-silvered mirror (beam splitter) M_s angled at 45º, which splits the light beam in two. One beam continues its path to mirror M_2, is reflected back toward M_s, where it is again reflected to the telescope.

The other beam is reflected to mirror M_1, where it is sent back through M_s straight to the telescope. The two beams recombine and form an interference pattern that is observed through the telescope.

At the laboratory of Hermann von Helmholtz in Germany, Michelson invented the interferometer based on a Maxwell idea proposed in 1875 to measure the earth's velocity relative to the ether. According to Maxwell, as the earth moves through space, the ether permeating space would create a wind. If we measure the velocity of light in the direction of the earth's movement around the sun, i.e., in the direction opposite the ether wind, like the boat traveling upstream (beam 2), the speed of light with respect to the ether would be equal to the value measured minus the speed of the ether wind, $c - v$. If we measure the speed of light in the direction of the ether (beam 2), i.e., downstream, its value would be the measurement obtained plus the speed of the ether, $c + v$. And if we measure the speed of light in the direction at right angles with the ether wind (beam 1), the situation would be analogous to the boat crossing the river. Figure 2-2 above illustrates these scenarios.

In the interferometer experiment, a light beam is sent to a beam splitter, the half-silvered mirror M_s angled at 45° degrees, which splits it in two: one beam is transmitted to the fixed mirror M_2, which reflects it back to M_s, where it is reflected toward the telescope. The other beam is reflected to the movable mirror M_1, which returns it to M_s, where it is transmitted on to the telescope. Eventually the two beams unite and interfere. As in the boat analogy, the two equal-length round trips of beam 1 (from M_s to M_1 and back to M_s) and beam 2 (from M_s to M_2 and back to M_s) should take different amounts of time, which can be detected in the interference pattern observable with the telescope. In his first attempt, **Michelson found no difference in the speed of light**, which to him was an unsuccessful result.

In 1887, Michelson now at Case Western Reserve University tried the experiment again with his colleague Edward W. Morley, a professor of chemistry. They set up their interferometer on a square slab of stone floating on mercury to minimize vibrations when the device was rotated in different directions. Different interference patterns would result from orienting the instrument along the direction of the earth's motion or across it. When the interferometer was rotated 90º, the arm along with the light beam that was parallel to the earth's velocity would become perpendicular to it, and the other arm and beam that were perpendicular would become parallel. Thus a shift in the interference pattern would be expected.

First, let us consider beam 2, which is parallel to the ether wind in Figure 2-3. Using the boat analogy of Figure 2-2, as beam 2 travels the distance l_2 downstream from M_s to M_2, its speed is $c + v$ and the time taken is $t = l_2 / (c + v)$. On its return trip from M_2 to M_s, its speed is $c - v$, and the time $t = l_2 / (c - v)$. Thus, beam 2's total time is:

$$t_2 = l_2 / (c + v) + l_2 / (c - v)$$

$$t_2 = 2l_2 / (c^2 - v^2) = 2l_2 / c(1 - v^2 / c^2).$$

Now, let us consider beam 1, which travels across the ether wind, from M_s to M_1 and back for a total distance of 2 l_1. Using the boat analogy in Figure 2-2, beam 1's motion forms a right triangle of sides c, v, and v'. From the Pythagorean theorem, we get beam 1's speed, $v' = \sqrt{c^2 - v^2}$. Beam 1's total time is thus:

$$t_1 = 2l_1 / \sqrt{c^2 - v^2} = 2l_1 / c \sqrt{1 - v^2 / c^2}$$

Assume that l_1 and l_2 are equal. Then beam 1lags behind beam 2 by an amount of time:

$$\mathbf{\Delta t = t_2 - t_1 = 2l_2 / c\,(1 - v^2 / c^2) - 2l_1 / c \sqrt{1 - v^2 / c^2}} \quad \textbf{(13)}$$

$$\Delta t = 2l / c\,[1 / (1 - v^2 / c^2) - 1 / \sqrt{1 - v^2 / c^2}\,]$$

This relation can be simplified if we use the binomial expansions, keeping only the first two terms:

$$1 - (v^2 / c^2)^{-1} \approx 1 + v^2 / c^2$$

and

$$1 - (v^2 / c^2)^{-1/2} \approx 1 + ½\,(v^2 / c^2)$$

$$\Delta t = 2l / c\,[(1 + v^2 / c^2) - (1 + ½\,(v^2 / c^2))] = l\, v^2 / c^3.$$

From this, $\Delta t = 0$ if $v = 0$ and the two beams will be in phase just as they were at the beginning. If $v \neq 0$, the two beams will be out of phase, and v could be determined. However, the earth cannot be stopped nor can l_1 and l_2 be independently assumed equal. At this point beam 1 is aligned to the arm l_1 and beam 2 is aligned to the arm l_2 of the apparatus, which is parallel to the ether wind.

To detect the difference in phase, Michelson and Morley rotated the interferometer through 90º so that beam 1 became parallel and beam 2 became perpendicular to the ether wind. In this rotated position, the roles of the beams were reversed, and the times (designated by primes) would be:

$$t'_1 = 2l_1 / c\,(1 - v^2 / c^2) \quad \text{and} \quad t'_2 = 2l_2 / c\,\sqrt{1 - v^2 / c^2}\,.$$

The time difference in the new position, thus becomes:

$$\boldsymbol{\Delta t' = t'_2 - t'_1 = 2l_2 / c\sqrt{1 - v^2 / c^2} - 2l_1 / c\,(1 - v^2 / c^2)} \qquad \textbf{(14)}$$

$$\Delta t' = 2l / c\,[1 / \sqrt{1 - v^2 / c^2} - 1 / (1 - v^2 / c^2)]$$

With the arms of the interferometer thus rotated, using relations (9) and (10), the fringe pattern will shift by:

$$\Delta t - \Delta t' = 2l_2 / c(1 - v^2 / c^2) - 2l_1 / c\sqrt{1 - v^2 / c^2} - 2l_2 / c\sqrt{1 - v^2 / c^2} - 2l_1 / c(1 - v^2 / c^2)$$

$$\Delta t - \Delta t' = 2 / c\,(l_1 + l_2)\,[1 / (1 - v^2 / c^2) - 1 / \sqrt{1 - v^2 / c^2}\,]$$

Again, if we assume $v / c \ll 1$, we can use binomial expansions to get:

$$\Delta t - \Delta t' \approx 2 / c\,(l_1 + l_2)\,[1 + (v^2 / c^2) - 1 - ½\,(v^2 / c^2)]$$

$$\boldsymbol{\Delta t - \Delta t' \approx (l_1 + l_2)\,(v^2 / c^3)} \qquad \textbf{(15)}$$

We take the speed of the earth around the sun (or equivalently the speed of the ether wind) $v = 3.0 \times 10^4$ m/s and the speed of light $c = 3.0 \times 10^8$ m/s. In the early Michelson-Morley experiments, each light beam was reflected many times by mirrors resulting in an effective length for each arm of about 11 m. Substituting these values in the relation (15), we obtain the time difference:

$$\Delta t - \Delta t' \approx 22 \text{ m } (3.0 \times 10^4 \text{ m/s})^2 / (3.0 \times 10^8 \text{ m/s})^3 \qquad (15)$$

$$\Delta t - \Delta t' \approx 22 \text{ m } (3.0 \times 10^{-16} \text{ m/s})$$

$$\Delta t - \Delta t' \approx 7.0 \times 10^{-16} \text{ s.}$$

With the center of the visible light spectrum at wavelength $\lambda = 5.5 \times 10^{-7}$m, its frequency would be $f = c / \lambda$:

$$f = (3.0 \times 10^8 \text{ m/s}) \;/\; (5.5 \times 10^{-7}\text{m})$$

$$f \approx 1.8 \times 10^{15}\text{s.}$$

$$f = 5.5 \times 10^{14} \text{ Hz.}$$

Thus, light wave crests would pass a point every $1 / (5.5 \times 10^{14} \text{ Hz}) \approx 1.8 \times 10^{-15}$s. Michelson and Morley would have found the time difference of 7.0×10^{-16} s to cause the interference pattern to shift by less than half a fringe:

$$7.0 \times 10^{-16} \text{ s} / 1.8 \times 10^{-15} \text{ s} \approx 0.4 \text{ fringe.}$$

Since their apparatus could detect fringe shifts as small as 0.01 fringe, the above shift should have been easily detected. Yet, *they found no significant shift in fringes*. They made observations with different orientations relative to the sun, experimented at different times of day and night, and in different seasons, and at different elevations. The result was still the same. There was no shift in the fringe pattern. Other experimenters also found the same negative result. Scientists scrambled to find an explanation. Never before had a null result of experiments caused so much concern and interest. One explanation proposed that the ether is at rest with respect to the earth and not to the sun and stars. But then it would mean the earth is somehow "preferred" over the sun and stars, contrary to the relativity principle. Another idea was that the ether was pulled along by the earth and other stars, and therefore had zero speed on their surfaces. But experiments with high-flying balloons at altitudes where the ether might be detected also found no traces of the ether wind.

The most notable hypothesis was advanced by the Irish physicist G. F. FitzGerald (1851-1901) in 1882. He proposed that the ether wind compresses all bodies that moved with it at the speed v, including the interferometer arm, the light beam, a meter stick, or any other object. Hence, this contraction makes it impossible to independently demonstrate the contraction. This shortening is by a factor exactly equal to the difference between the speed of light along the ether's direction and the speed of light across it, calculated to be $(1 - v^2 / c^2)^{1/2}$ or, since $\beta = v / c$, $(1 - \beta^2)^{1/2}$, which is the reciprocal of γ. In 1895 the Dutch physicist Hendrik A. Lorentz (1853-1928) independently proposed the same contraction in terms of changes in the electromagnetic forces between the atoms, but without providing empirical evidence. Now known as **the Lorentz-FitzGerald contraction**, the ad hoc proposal advanced merely to explain the Michelson-Morley null results is just a consequence of the special theory of relativity. The contraction, however, is real as we will see in Section 2.4.1 below. For his works with light, Michelson became the first American to win the Nobel prize in physics in 1907.

The conclusion drawn from the null results of the Michelson-Morley experiments is that **there is no ether** to make a difference in the time of travel of the two light beams along and across the ether. The speed of light is not in the least affected by the earth's motion. A 1979 version of the Michelson-Morley experiment using very stable lasers to improve the measurement precision by a factor of 4000 still found no traces of the ether wind.

Example 13. In a Michelson-Morley experiment, suppose each of the light beams has length l = 28 m. Given $v = 3.0 \times 10^4$ m/s and $c = 3.0 \times 10^8$ m/s, (a) what is the time difference caused by rotating the arms of the interferometer through 90º? (b) Assuming the wavelength of the light used is λ = 500 nm, what is the expected fringe shift?

Solution. (a) Applying the time difference formula (15), we have:

$$\Delta t - \Delta t' \approx (l_1 + l_2)(v^2 / c^3) \qquad (15)$$

$$\Delta t - \Delta t' \approx (28 \times 2) \text{ m } (3.0 \times 10^4 \text{ m/s})^2 / (3.0 \times 10^8 \text{ m/s})^3$$

$$\Delta t - \Delta t' \approx 56 \text{ m } (3.0 \times 10^{-16} \text{ m/s})$$

$$\Delta t - \Delta t' \approx \underline{19.0 \times 10^{-16} \, s}.$$

(b) First let us find the frequency of the light used, $f = c / \lambda$:

$$f = (3.0 \times 10^8 \text{ m/s}) \ / \ (5.0 \times 10^{-7} \text{m})$$

$$f \approx 1.7 \times 10^{15} \text{ s}.$$

Hence, the fringe shift is the time difference divided by the reciprocal of the frequency of the light beam:

$$19.0 \times 10^{-16} \text{ s} / 1.7 \times 10^{-15} \text{ s} = \underline{1.1 \text{ fringes.}}$$

We now know that evidence of the hypothesized ether wind does not exist, and Einstein's theory of relativity disposes of the concept entirely.

Example 14. (a) If a particle is moving at a speed of 0.2000 c, find the value of $\gamma = 1 / (1 - v^2 / c^2)^{1/2}$. (b) Calculate the γ of a particle moving at a speed of 0.0020 c. (c) Find the γ of a particle moving at a speed of 0.000978 c (roughly the speed of a commercial jetliner).

Solution. (a) We substitute the particle's speed v in the γ formula:

$\gamma = 1 / (1 - v^2 / c^2)^{1/2}$

$\gamma = 1 / [1 - (0.2000\, c)^2 / (c^2)]^{1/2} = 1 / (1 - 0.0400)^{1/2} = 1 / 0.9798 =$ *1.021.*

(b) We apply the same procedure:

$\gamma = 1 / (1 - v^2 / c^2)^{1/2}$

$\gamma = 1 / [1 - (0.0020\, c)^2 / (c^2)]^{1/2} = 1 / (1 - 0.0000040)^{1/2} = (1 / 0.999996)^{1/2} =$ *1.000002.*

(c) The same procedure yields:

$\gamma = 1 / (1 - v^2 / c^2)^{1/2}$

$\gamma = 1 / [1 - (0.000978\, c)^2 / (c^2)]^{1/2} = 1 / (1 - 0.000000956)^{1/2} = (1 / 0.9999990435)^{1/2} =$ *1.000000478.*

Or using the approximation of γ derived by binomial expansion, we get:

$$\gamma \approx 1 + \beta^2 / 2 \qquad (7)$$

$\gamma \approx 1 + (0.000978\, c^2 / c^2)^2 / 2 = 1 + 0.000000478 =$ *1.000000478.*

If we compare the results of (a) and (c), we can see the effects of speed on relativistic length and everyday length, in the measurement of which γ is the only factor that separates them. Recall that in the Galilean transformation, $x = x' + vt'$ and in the Lorentz transformation $x = \gamma\,(x' + vt')$. At a high speed of 0.20 c, γ is fairly significant at 1.021 while at a slower jet plane speed, a γ of 1.000000478 is so close to 1 that its effect on length is negligible. Table 2-1 shows that as v gets much closer to c, for example, $v = 0.999\,999\, c$, γ becomes huge, $\gamma = 707.107$.

The factor γ is important in relativity measurements. As shown in Table 2-1 above, since $\beta = v / c$ is never zero (except when the object is at rest) although it is very small in everyday experience, $\gamma > 1$ whereas its reciprocal is less than 1, $1 / \gamma < 1$.

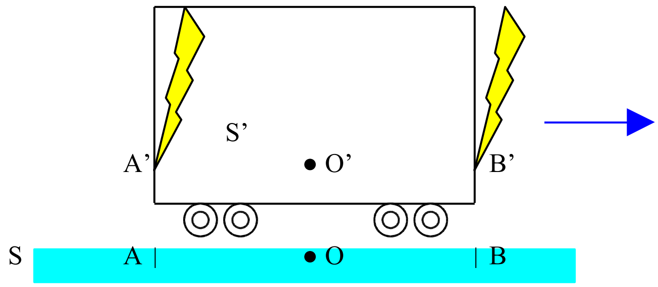
2.2 Simultaneity

Basic to Newtonian mechanics is the premise of universal time, which is the same for all observers, independent by nature of anything external to it. This conforms to our everyday common sense experience. We are used to believing that an hour is equal to an hour, whether we are in California or on the Moon. However, we know that Einstein had to abandon the notion of fixed time in his theory of relativity. According to this theory, measurements of time change depending on reference frames.

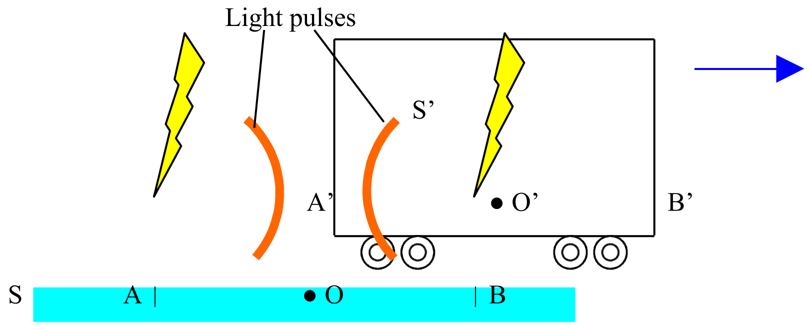
Einstein proposed a thought experiment to prove his point. A boxcar traveling to the right at constant velocity is struck by two bolts of lightning that leave marks on each end of the boxcar and on the ground (Figure 2-4). The marks on the boxcar are A' and B', and the corresponding marks on the ground are A and B. In the moving boxcar (frame S') an observer O' stands halfway between A' and B'. On the ground the stationary observer O (frame S) is halfway between A and B.

Suppose the light pulses from the lightning bolts reach the midway ground observer O simultaneously. (Figure 2-4). The observer O says that she sees the two bolts striking at A and B at the same time, and concludes that *the two events are simultaneous*. But with the moving observer O' things are different. By the time the light pulse has reached the observer O, the boxcar has moved a distance to the right toward the front bolt, carrying the observer O' with it, and thus shortening the distance between her and the front light pulse. The light pulse from B' has already passed O' while the pulse from A' has not yet arrived. And this is because as the boxcar moves to the right, it gets closer to the lightning bolt at the front and recedes from the one in the rear. Since the speed of light is constant, light takes less time traveling a shorter distance. The observer O', therefore, sees the lightning bolt at the front of the car strike first and the lightning bolt at the rear strike next. Thus *to O' the two events are not simultaneous*.

We conclude that **events that are measured to be simultaneous in one frame of reference are not simultaneous in another frame**.



(a)



(b)

Figure 2-4. (a) To a midway stationary observer O on the ground (frame S), the two lightning bolts appear *to strike simultaneously at points A and B*. (b) To the midway observer O’ in the train moving to the right (frame S’), the lightning bolt at the front of the car strikes first, before the lightning bolt in the back strikes. *The two events A’ and B’ are not simultaneous*.

This thought experiment shows that **events that are simultaneous in one reference frame are not simultaneous in another reference frame even though both are inertial frames**. Hence, just as there is no such thing as absolute motion or absolute rest, **there is no such thing as absolute simultaneity**. But **there is relative simultaneity**, i.e., relative to reference frames. This is a direct consequence of the second postulate.

Along this same line of investigation, let us take another example. It shows that if two events are simultaneous for an inertial observer located halfway between them, they may not be simultaneous for all midway inertial observers regardless of their motion.

In Figure 2-5, a stationary space station is commanded by Commander Achilles while a rocket ship moving in the opposite direction is under Commander Helen. We will arrange, at the instant that Commander Achilles and Commander Helen are opposite each other (Figure 2-5a), for two flares to go off at the same time. Flare A makes a dent mark A on Achilles’ station and A’ on Helen’s rocket ship. Flare B makes a dent mark B on Achilles’ station and B’ on Helen’s ship.

As a midway observer, Achilles saw the flashes from Flare A and Flare B occur in the same instant. Hence, he concluded that the two flashes were simultaneous events. To him, the length AB was equal to the length A’B’. As he looked out his window, he saw Helen’s ship sliding off to his station’s rear, moving toward Flare A and away from Flare B while his station was at rest. He thus saw that Helen reached the pulse from Flare A first before the pulse from Flare B reached her. It was obvious to Achilles that Helen should see what he saw, i.e., Flare A fired first, and Flare B fired next. The flares did not go off at the same time.

For her part as a midway observer, Helen considered herself being stationary and saw Achilles’ station move off to the rear of her ship. Helen observed the flash from Flare A occur first, and Flare B fired second as the station was moving toward the latter. Thus she espied that the points A and A’ coincided first before the points B and B’ did. To her two events were never simultaneous, and the length AB was not equal to A’B’.

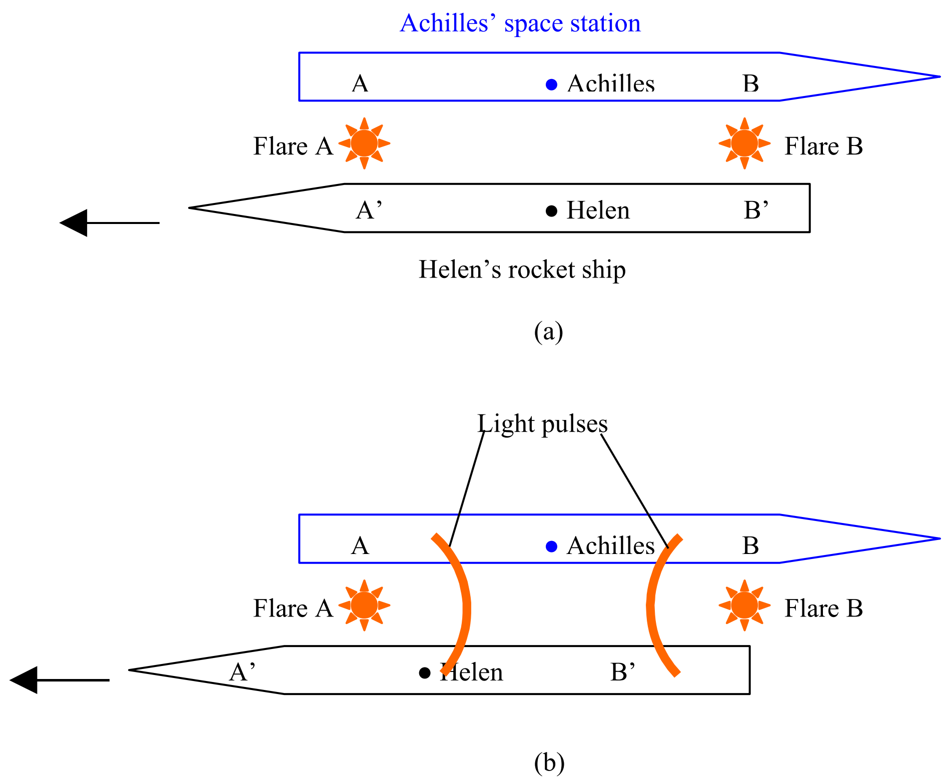


Figure 2-5. (a) Achilles, a midway observer in his stationary space station, observes that Flares A and B go off at the same time and AB = A'B'. But he sees the flash from Flare A reach Helen's rocket ship first as it moves toward the station's rear. (b) As a midway observer, Helen sees Achilles' station sliding off toward her ship's rear, and observes that the pulse from Flare A occurs before that from Flare B. To Helen, AB ≠ A'B'. Both observers are correct by the first postulate of the special theory of relativity.

Neither Achilles nor Helen would agree on lengths, times, or simultaneity. Yet by the first postulate of the special theory of relativity, they are both correct. There is no preferred reference frame. Is there any way to reconcile their different observations? No. It is in the nature of things, the way the universe is. It is reality.

Synchronization of Clocks

To determine time and simultaneity between two events, it is necessary to **synchronize the clocks.** It will not do to synchronize two clocks in the same location, then have one clock travel while keeping the other clock stationary. As we will see in the next section, the **time dilation** effect will make the traveling clock run slower than the one at rest.

We will follow a procedure proposed by Einstein that uses the speed of light to synchronize all the clocks within the same frame of reference. A flash bulb is placed midway between two clocks properly spaced in a reference frame. When the flash is set off, light travels with velocity c to both clocks. By traveling the same distance in opposite directions light reaches the clocks at the same time. When the light pulses reach the clocks, an observer at each clock sets it to the same time. The same procedure can be repeated to synchronize all the clocks within the same reference frame. With this series of synchronized clocks, we can judge the simultaneity of two events that are widely separated in space. Thus, besides being a constant of nature, light is inseparably related to time and simultaneity.

Another way to synchronize identical clocks also uses light. We need two observers, each one stationed at her clock in the same frame S. We could use as many observers as there are clocks to set. For simplicity, the first observer is at the origin O and the other at a known distance x from O (Figure 2-6).

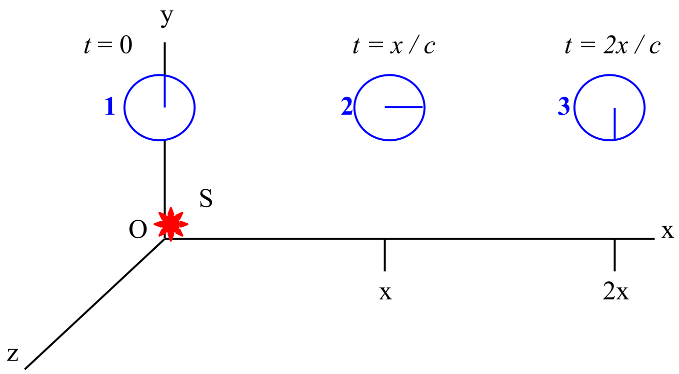


Figure 2–6. Identical clocks are evenly spaced at known distances *x* from O in stationary frame *S*. A flash of light is set off at O. At that instant the observer at O sets her clock 1 to $t = 0$. When the light pulse reaches point x, the observer at x sets her clock 2 to $t = x / c$. Likewise the observer at point 2x sets her clock 3 to $t = 2x / c$ at the pulse's arrival.

A flare flashed by observer O at $t = 0$ travels at light velocity. Observer O sets her clock 1 to $t = 0$. When the light pulse arrives at clock 2 at point x, the second observer sets it to $t = x / c$, this being the time light took to travel the distance *x*. A third clock located at 2x is set in the same manner. By using this procedure, identical clocks in the frame *S* can be successively set and they are all synchronized. The same process can also be applied to synchronize identical clocks in a moving frame *S'*. Now suppose we want to set an array of clocks in a moving frame *S'* to the original clock in frame *S*. At time $t = 0$, both frames coincide so that their origins O and O' are superposed. We synchronize the clocks at the origins to that time. When the observer in frame *S* sets her clock to time $t = 0$, her counterpart in frame *S'* sets her clock to $t' = 0$. Then when an event occurs at *x* in frame *S*, the observer in *S'* records the same event at point x' and time $t' = x' / c$. Now a problem arises since the time of a clock in *S'* depends not only on time *t* of a clock in *S* but also on its position in that frame. The greater is the distance of this clock from its origin in *S*, the greater is the discrepancy between *t* and *t'*. This is a consequence of the constancy of the speed of light (the second postulate). Time is relative, and not fixed as Newton would have it. So too is simultaneity relative.

If in everyday life we are not aware of all this relativity, it is because its effects are noticeable only when the relative speed between reference frames is very large (close to *c*) or when the frames are separated by vast distances.

2.3 Time Dilation

We have seen that one of the consequences of the special theory of relativity is that **simultaneity of events is relative to the observer's frame of reference.** Events that occur in the same place and are simultaneous in one inertial frame are not simultaneous in another. Yet another consequence of the theory is a very counterintuitive phenomenon of time dilation, time that is not fixed or absolute but runs slow depending on frames of reference.

2.3.1 Time Dilation

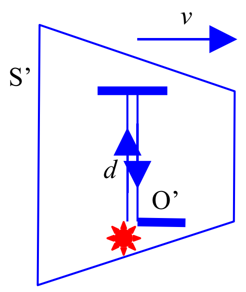
In Newtonian physics time is absolute and exists independently of anything outside it. This corresponds to our concept of time, which we consider distinct from space and not ordinarily a "dimension." Besides, we do not easily accept the notion of time as being relative because we have never experienced it. But just like any revolutionary advances in science, the theory of relativity upsets old notions and introduces new perspectives about the reality. To see what effects this theory has on our measurement of time, we will as usual consider a thought experiment with a light clock as depicted in Figure 2-7.

Traveling in a spaceship (frame *S'*) moving at high speed *v* to the right, observer O' flashes a light. The light pulse travels to the mirror fixed on the ship's ceiling at a distance *d* from the ship's floor. Observer O' measures the round trip of the light pulse (to the mirror and back to the detector) with an accurate clock, using the definition of the velocity (Figure 2-7a). Since light travels at c, the round trip takes a time:

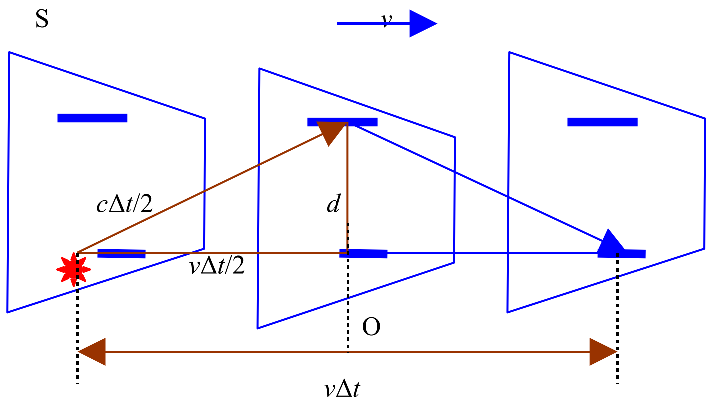
$$\Delta t' = \text{distance traveled} / \text{velocity} = 2d / c.$$

The distance traveled is:

$$d = c\Delta t' / 2.$$



(a) Light pulse sent by moving observer O' travels straight up and down covering the distance d twice. Time traveled to mirror and back: $\Delta t' = 2d / c$.



(b) Stationary observer O in frame S on Earth sees the same light pulse take a diagonal path, not the vertical distance d, and thus taking a longer time Δt to complete the round trip.

Figure 2-7. (a) A flash bulb set off in a spaceship traveling at the speed v in frame S' is reflected off a mirror fixed on the ceiling at a distance d to reach a detector by the observer O' at the light source. The light's up-and-down round trip takes time $\Delta t' = 2d / c$. (b) To a stationary observer O on earth (frame S), however, the light's path is a zigzag as shown, i.e., a longer distance than d. Therefore, the round trip time traveled Δt in frame S is greater than $\Delta t'$ in frame S'.

The time interval Δt' between two events that occur at the same location measured by a single clock, as in frame S', is called the **proper time.** **Proper** means characteristic of or belonging to. The proper time is usually denoted as Δt_0, which is the notation we use from here on. *The proper time is the shortest time interval measured with a single clock by any observer*. We have the proper time formulation:

$$\boldsymbol{\Delta t_0 = 2d / c} \qquad \textbf{(16)}$$

from which we derive **the distance traveled**:

$$d = c\Delta t_0 / 2.$$

In the context of a light clock Δt_0 makes up one tick. Meanwhile the stationary observer O in frame *S* of the earth, equipped with an accurate clock, observes the same process and notices that the light's path is not straight up and down. By the time the light reaches the mirror on the ceiling, the spaceship has moved a certain distance. As the light is reflected down to reach the detector the spaceship has again moved a comparable distance, as shown in Figure 2-7b, covering a total distance of $v\Delta t$ (from the definition of velocity). For its part, the light pulse will have traveled, during the same time, a distance of $c\Delta t$. As can be seen in Figure 2-7b, the distance covered by light to the mirror is the hypotenuse of a right triangle whose other sides are the distance traveled by the spacecraft and the distance *d* from the mirror to the detector. Applying the Pythagorean theorem yields:

$$(c\Delta t / 2)^2 = d^2 + (v\Delta t / 2)^2$$

Since $d = c\Delta t_0 / 2$,

$$(c\Delta t / 2)^2 = (c\Delta t_0 / 2)^2 + (v\Delta t / 2)^2$$

$$(c^2\Delta t^2 / 4) - (v^2\Delta t^2 / 4) = c^2\Delta t_0^2 / 4$$

$$c^2\Delta t^2 - v^2\Delta t^2 = c^2\Delta t_0^2$$

Solving for Δt, we get:

$$\Delta t^2 \ (c^2 - v^2) = c^2\Delta t_0^2$$

$$\Delta t^2 = c^2\Delta t_0^2 / (c^2 - v^2)$$

$$\Delta t = c\Delta t_0 \ / (c^2 - v^2)^{1/2}$$

$$\Delta t = c\Delta t_0 \ / c(1 - v^2 / c^2)^{1/2}$$

Time dilation formula:

$$\boldsymbol{\Delta t = \Delta t_0 \ / \ (1 - v^2 / c^2)^{1/2}} \qquad \textbf{(17)}$$

$$\Delta t = \gamma \, \Delta t_0 \qquad (18)$$

The spreadsheet formula of time dilation relation (17) is:

$$\boldsymbol{\Delta t = \Delta \mathrm{t_0/(1-(v/c)\char`^2)\ \char`^(1/2)}} \qquad \textbf{(19)}$$

Substitute actual values for Δt_0, v, and c in the formula 19 to get the result. If v is expressed in terms of c, e.g., 0.95 c, then this v value takes the place of v/c in the formula.

The **proper time** formula is:

$$\Delta t_0 = \Delta t / \gamma \qquad (20)$$

$$\boldsymbol{\Delta t_0 = \Delta t \, (1 - v^2 / c^2)^{1/2}} \qquad \textbf{(21)}$$

The spreadsheet formula of the proper time relation (21) is:

$$\boldsymbol{\Delta \mathrm{t_0} = \Delta \mathrm{t{*}(1-(v/c)\char`^2)\ \char`^(1/2)}} \qquad \textbf{(22)}$$

Substitute actual values for Δt, v, and c in the formula 22 to get the result. If v is expressed in terms of c, e.g., 0.95 c, then this v value takes the place of v/c in the formula.

Hence, the factor $\boldsymbol{\gamma}$ in terms of the two time intervals Δt and Δt_0:

$$\boldsymbol{\gamma = \Delta t \ / \ \Delta t_0} \qquad \textbf{(23)}$$

To find **the speed of travel *v*** in terms of Δt and Δt_0, we start from the above formula:

$$\gamma = \Delta t \ / \ \Delta t_0 \qquad (23)$$

$$(\Delta t \ / \ \Delta t_0)^2 = (1 - v^2 / c^2)^{-1}$$

$$\Delta t^2 = (1 - v^2 / c^2)^{-1} \ \Delta t_0^2$$

$$\Delta t_0^2 = (1 - v^2 / c^2) \ \Delta t^2$$

$$\Delta t_0^2 \ / \ \Delta t^2 = (1 - v^2 / c^2)$$

$$v^2 / c^2 = 1 - \Delta t_0^2 \ / \ \Delta t^2$$

$$\mathbf{v = (1 - \Delta t_0^2 \ / \ \Delta t^2)^{1/2} \ c} \qquad \mathbf{(24)}$$

In the time dilation formula (17) above note that the denominator is the reciprocal of the factor γ, which recurs in relativistic measurements. Since γ and its reciprocal $1 / \gamma$ are ubiquitous in relativistic measurements, we need to learn to manipulate them with ease. The time dilation formula indicates that the time in the moving frame *S'* or the *proper time* Δt_0 is shorter than the time Δt in the stationary frame *S* because $\gamma > 1$ (Table 2-1 above). In other words, **time runs more slowly in a moving body than in a body at rest by exactly the factor *γ***. This phenomenon is not due to any clock's accuracy; it is simply the nature of time, i.e., **time is relative**, a reality that is captured elegantly by the special theory of relativity. We call this slowing down of time **time dilation.**

What is the meaning of γ? From one perspective, γ is the factor by which the proper time differs from the time measured of two events in two different reference frames. Thus, γ is the ratio $\gamma = \Delta t / \Delta t_0$ of the time interval between two events occurring **in different frames** as measured by a clock at rest to the proper time, i.e., the time interval measured of two events occurring **at the same location** by an inertial observer using a single clock. The factor γ is, in effect, the amount of time Δt corresponding to every unit of proper time Δt_0. For instance, if $\Delta t_0 = 1$ s, $\Delta t = \gamma$ s. The factor γ is speed-sensitive. In Table 2-1, we see that as the speed ratio $\beta = v / c$ increases, γ increases. For example, when $v = 0.4\ c$, $\gamma = 1.09$, and when $v = 0.5\ c$, $\gamma = 1.15$.

Time dilation as a concept is not easy to grasp by our common sense because we are so used to thinking of time as absolute. Yet it has been corroborated by experimental evidence. Let us examine the time dilation formula in further detail.

If the speed of a moving object v is 0, then both times are equal, $\Delta t = \Delta t_0$ as expected. In other words, when nothing moves, there is no time dilation.

If $v > 0$ but $v << c$ as in everyday experience, then $\gamma \approx 1$, and time dilation has no observable effect, i.e., it is not measurable by ordinary clocks. Only when v is closer to c as in certain experiments already conducted, in future intergalactic travels or in particle accelerators does time dilation have the measurable effects spelled out in formula (17).

Time dilation can be measured by using clocks that are accurate to the nanoseconds.

Experimental confirmation of time dilation was achieved in October 1971 the physicists J. C, Hafele and R. E. Keating placed four high-precision cesium-beam atomic clocks (accurate to the nanoseconds) on commercial airplanes flown around the world twice, in opposite directions, and identical clocks at the U.S. Naval Observatory. They found that after flying around the world at high speed the flying clocks ran slower than the stationary clock in the lab. From time dilation calculations the eastward clocks were expected to have lost 40 ± 23 ns in the trip whereas the westward clocks should have gained 275 ± 21 ns. The experimental results were in remarkable agreement with theory: The eastward clocks lost 59 ± 10 ns and the westward clocks gained 273 ± 7 ns! Due to the relativistic effects of time dilation atomic clocks that are moved from one location to another must now be adjusted for these effects. However, in everyday activities, we have no clocks that are accurate enough to measure time dilation.

Knowing that $\gamma = (1 - v^2 / c^2)^{-1/2}$ (Formula 3), and noting also that $\gamma = \Delta t / \Delta t_0$ (Formula 23) as derived from the time dilation equation, we work out a few data points that relate β and γ for the graph in Figure 2-8 below, which shows time dilation as a function of speed v, i.e., γ as a function of β:

Time dilation as a function of speed *v*

$\beta = v / c$	$\gamma = (1 - v^2 / c^2)^{-1/2}$
0.1	1.005037815
0.2	1.020620726
0.3	1.048284837
0.4	1.091089451
0.5	1.154700538
0.6	1.250000000
0.7	1.400280084
0.8	1.666666667
0.9	2.294157339
0.99	7.088812050

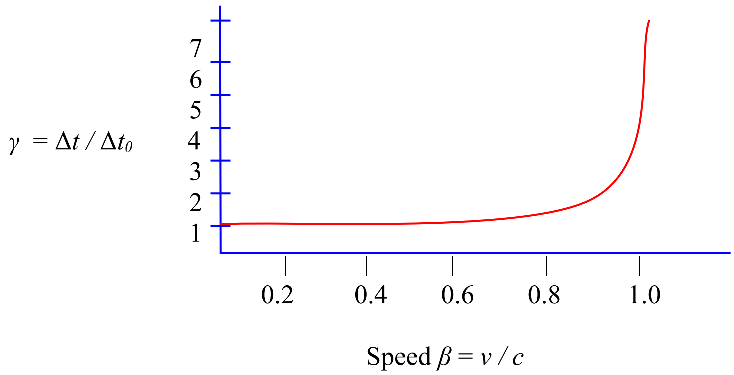


Figure 2-8. Relationship between γ and β or time dilation as a function of speed.

The data above shows a very gradual rise in the value of $\gamma = \Delta t / \Delta t_0$, starting from a v of 10% of the speed of light. So far we do not have technology to produce any machine that can move that fast. The time curve (γ) hardly changes for speeds v up to 0.2 c (most everyday speeds are well below this), and rises moderately until v reaches 0.9 c, when $\gamma = \Delta t / \Delta t_0$, more than doubles its value. The last two data points give a hint of the rapid rise of γ as $\beta = v / c$ increases in significant decimal places. From this point on, the rise of γ is dramatic. Still the γ curve will never reach 1.0 on the x-axis because c is the unreachable upper limit of all speeds.

The graph in Figure 2-8 can be generalized as γ as a function of speed v / c. This function recurs in all relativistic calculations.

In a time dilation problem, three important values are involved: (1) *the proper time* $\boldsymbol{\Delta t_0}$ between two events occurring at the same physical location moving at (2) *a speed* $\boldsymbol{v}$ *relative to a stationary reference frame*, and (3) *the time* $\boldsymbol{\Delta t}$ *(sometimes called stationary time*) between these events as measured in the stationary frame. If any two of these values are known, the third one can be derived in a straightforward manner from the time dilation equation. The fourth value is a given: the speed of light $\boldsymbol{c}$.

Let us illustrate time dilation with a few examples.

Example 15. A light clock travels in a spaceship with a speed of 0.600 c relative to a stationary observer on the earth. How long does it take for the traveling clock to advance by 1 second according to this observer?

Solution. First we determine the proper time Δt_0, the time elapsed between two events that take place in the same location. In this case, it is the 1.0 second recorded by the moving light clock in the spaceship. We are to find the time Δt as measured by the observer on the earth. Applying the time dilation formula (17), we get:

$$\Delta t = \Delta t_0 \,/\, (1 - v^2 / c^2)^{1/2} \qquad (17)$$

$$\Delta t = 1.00 \text{ s} / [1 - (0.600 \text{ c})^2 / c^2]^{1/2}$$

$$\Delta t = 1.00 \text{ s} / (0.640)^{1/2} = 1.00 \text{ s} / 0.800 = \underline{1.25 \text{ s.}}$$

Example 16, Find the factor by which a clock on the space shuttle traveling at 7599 m/s runs slower than an identical clock on the earth.

Solution. Recall that γ is the factor by which Δt differs from the proper time Δt_0 as indicated in the formula 14, $\Delta t = \gamma \Delta t_0$. We find γ as follows:

$$\gamma = (1 - v^2 / c^2)^{-1/2} \quad (3)$$

$$\gamma = [1 - (7599 \text{ m/s})^2 / (3.00 \times 10^8 \text{ m/s})^2]^{-1/2}$$

$$\gamma = \underline{1.000000000320800.}$$

For every second recorded by the clock on the space shuttle, the clock on the earth records 1.000000000320800 seconds. Obviously no ordinary clocks can measure this time. Hence, *the time dilation effect is totally negligible at the shuttle speed, as it is in all speeds we encounter in everyday experience.*

Example 17, The starship Encounter travels with the speed of 0.70 c relative to the earth for 15 years according to the ship's clock. To an observer on the earth how many years elapse during the trip?

Solution. We determine that the proper time Δt_0, the time measured by a clock at rest on board the ship, is 15 years. By applying the time dilation formula $\Delta t = \Delta t_0 / (1 - v^2 / c^2)^{1/2}$, we find the earth duration to be:

$$\Delta t = \Delta t_0 / (1 - v^2 / c^2)^{1/2} \quad (17)$$

$$\Delta t = 15 \text{ y} / [1 - (0.70^2 c)^2 / c^2]^{1/2} = 15 \text{ y} / (1 - 0.70^2)^{1/2}$$

$$\Delta t = 15 \text{ y} / 0.51^{1/2} = \underline{21.00 \text{ y.}}$$

Example 18, Astronaut Daedalus travels with a speed of 0.997 c to Galaxy Minos 20 ly (light-years) away then turns back to Earth. His son Icarus, age 20 when he left, comes to Starport to greet him. (a) How old is Daedulus on his return to Earth if he was 45 years old when he set out on his journey? (b) How old is Icarus when he sees his father again?

Solution. As in all time dilation problems, we first determine two events, and the proper time. The two events are Daedalus' departure from and return to Earth. The proper time is the time Daedalus measures during his 20-ly voyage. First we find the duration of Daelalus' travel from the Icarus' point of view. Knowing that the distance Daelalus traveled is 40 ly (round trip), i.e., a distance covered by light for 40 years or 40 c (1 y), at the velocity of 0.997 c, the time it took Daelalus as measured from Earth is:

$$\Delta t = d / v$$

$$\Delta t = 40\, c \text{ (1 y)} / 0.997\, c = 39 \text{ y}$$

(a) Daedalus, however, took the proper time Δt_0 to make the journey:

$$\Delta t_0 = \Delta t\, (1 - v^2 / c^2)^{1/2} \quad (21)$$

$$\Delta t_0 = 39 \text{ y } [1 - (0.997^2 c)^2 / c^2]^{1/2}$$

$$\Delta t_0 = 39 \text{ y } (0.005991)^{1/2} = 3.02 \text{ y}$$

From his point of view, Daedalus had traveled only 3.02 years. Therefore, at the end of the journey, he is:

$$45 \text{ y} + 3.02 \text{ y} = \underline{48.02 \text{ years old}}.$$

(b) His son Icarus, who stayed back on Earth, is 39 years older:

$$20 \text{ y} + 39 \text{ y} = \underline{59 \text{ years old.}}$$

Thus the father is younger than the son after 39 years of absence!

Example 19, Captain Valiant embarked at very high speed on an interstellar odyssey that lasted 12 years by Earth's clock. When he came home, he had aged 2 years. What was his average speed?

Solution. We know the two events, his departure and his return. The time measured with respect to Earth is $\Delta t = 12$ y. But Captain Valiant's proper time is only $\Delta t_0 = 2$ y. We apply the speed formula:

$$v = (1 - \Delta t_0^2 \,/\, \Delta t^2)^{1/2}\ c \qquad (24)$$

$$v = (1 - 2^2 \text{ y} / 12^2 \text{ y})^{1/2}\ c = \underline{0.986\ c.}$$

Example 20, Mycenaen Commander Agamemnon's spaceship takes him on a long trip to Galaxy Troy with velocity of 0.75 c, and brings him back to Earth. When he checks the clock and calendar in his house, he finds that a period of seven years has elapsed. How long did his trip take?

Solution. Again the two events are his departure and his return. Earth time duration is $\Delta t = 7$ y. We are to find the proper time Δt_0 by using the proper time formula (21):

$$\Delta t_0 = \Delta t\ (1 - v^2 / c^2)^{1/2} \qquad (21)$$

$$\Delta t_0 = 7 \text{ y}\ (1 - 0.75^2\, c^2 / c^2)^{1/2}$$

$$\Delta t_0 = 7 \text{ y}\ (1 - 0.5625)^{1/2} = 7 \text{ y}\ (0.4375)^{1/2} = \underline{4.63 \text{ y.}}$$

Example 21, A spacecraft reenters Earth at the speed of 15 km/s and takes 30 seconds to land. What is the time dilation effect on the spacecraft during this time?

Solution. The two events are the spacecraft's reentry and touchdown. The proper time Δt_0 for the spacecraft is $\Delta t_0 = 30$ s. If we know γ, we can derive the effect Δt by applying the time dilation formula:

$$\Delta t = \Delta t_0 \,/\, (1 - v^2 / c^2)^{1/2} \qquad (17)$$

First we convert v into a fraction of c for ease of handling:

$$v = 15 \text{ km/s} / 3 \times 10^6 \text{ km/s} = 0.00005\ c$$

Applying the time dilation formula yields:

$$\Delta t = 30 \text{ s}\ / (1 - 0.00005^2\, c^2 / c^2)^{1/2}$$

$$\Delta t = 30\text{s}\ / (0.9999999975)^{1/2} = 30 \text{ s} \times 1.000\,000\,001\,25 = \underline{30.000\,000\,38 \text{ s.}}$$

Even at this unrealistically high speed, the effect of time dilation is so tiny it is undetectable by an ordinary clock.

Example 22, Cosmonaut Svoboda in a spacecraft hurtling toward Galaxy Tranto Mir measures her heartbeat to be one every 0.90 s while Mission Control on Earth measures her heart to beat once every 1.50 s. How fast is Svoboda traveling relatively to Earth?

Solution. We find the two events in the beginning and end of one of Cosmonaut Svoboda's heartbeat. Since the heartbeat begins and ends in the same location, her heart, the heartbeat she measured is the proper time Δt_0. Knowing the time Δt measured in the earth's reference frame (Mission Control), we apply the speed relation derived from the time dilation formula:

$$v = (1 - \Delta t_0^2 \ / \ \Delta t^2)^{1/2} \ c \qquad (24)$$

$$v = (1 - 0.90^2 \text{ s} / 1.50^2 \text{ s})^{1/2} \ c = \underline{0.80 \ c.}$$

Example 23, An experiment conducted in a laboratory on Earth takes 300 s to complete. A Martian space traveler moving with a constant velocity of 0.6 *c* relative to Earth happens to observe the experiment. How long does the Martian measure the experiment to last?

Solution. The two events are the beginning and end of the lab experiment. Since both events occur in the lab, the time measured is the proper time $\Delta t_0 = 300$ s. We find the time Δt measured by the Martian to be:

$$\Delta t = \Delta t_0 \ / \ (1 - v^2 / c^2)^{1/2} \qquad (17)$$

$$\Delta t = 300 \text{ s} \ / \ (1 - 0.6^2 \ c^2 / \ c^2)^{1/2}$$

$$\Delta t = 300 \text{ s} \ / \ (1 - 0.36 \ c^2 / \ c^2)^{1/2} = 300 \text{ s} / 0.8 = \underline{375 \text{ s.}}$$

What are the implications of relativistic time dilation?

The concept of time dilation implies that as Δt goes to infinity, one tick of the clock (or any one unit of time) takes an infinite amount of time assuming that Δt_0 is one such tick. To put it another way, as *v* approaches *c* the clock gradually grinds to a halt. That is another sense of *c*'s being the *upper limit of all speeds.*

Another way to look at time dilation is that the proper time Δt_0, the time elapsed between two events that occur at the same location, e.g., the time traveled by an astronaut, remains small while $\gamma = (1 - v^2 / c^2)^{-1/2}$ increases very rapidly as the speed *v* approaches, but never reaches, *c*. The relation $\Delta t = \gamma \, \Delta t_0$ derived from the time dilation equation says that the time Δt elapsed on Earth is equal to the proper time Δt_0 multiplied by γ. That is, every second of proper time that an astronaut travels near the speed of light corresponds to γ seconds as measured by an earthbound observer. As γ grows by leaps and bounds even with tiny $\beta = v / c$ increases, so also do the time dilation effects. Thus, as shown by Table 2-3 in Section 2.1.2 above, if the astronaut travels with a velocity of 0.9994 *c*, for every year (second, minute, hour, and so on.) she clocks in her spaceship, her brother on the earth clocks 28.87 years (seconds, minutes, hours, and so on). And if she travels at velocity 0.9999 *c* for a year, her brother will age 70.71 years, an extremely tantalizing consideration for attempting interstellar travel.

Another example of space travel made possible, but highly implausible, by time dilation. A spaceship that can accelerate at a constant and comfortable rate of *g* (acceleration of Earth's gravity = 9.80665 m/s^2) can take travelers to Andromeda galaxy 2 million light-years away and back in 59 years. During this time the earth has aged more than 4 million years. And if the ship travels 78 years with the same acceleration, it can reach a galaxy 500 million light-years away to find upon return that the earth has become one billion years older. This is the law of nature. But as with other laws of nature, we do not have the technology or the know-how yet to take full advantage of them.

One caveat about space travel, however. It takes an enormous amount of energy, in fact, an infinite amount of energy, in order to build up enough acceleration to reach this near light speed, an engineering feat that our technology is not capable of achieving within the foreseeable future. To add a little perspective, NASA's space shuttle, which in 2005 is the fastest manned flying machine available, needs 3 space shuttle main engines (SSME's built into the orbiter), which are fueled by the an external tank (ET) of propellant, and 2 reusable solid rocket boosters (SRB's) burning solid propellant to lift it free of Earth's gravitational pull for orbit insertion. The 29,937-kg external tank with its 719,115-kg load of liquid oxygen and liquid hydrogen weighs a total of 749,052 kg. Each of the two 86,137-kg SRB's holds 494,064 kg of rocket propellant, weighing a total of 580,201 kg. Together the 2 SRB's deliver 71.4% (or 2,993,740 kg) of the total thrust to the shuttle at liftoff and during first stage ascent. At liftoff the ET receives a total thrust of 3,538,057 kg from the 2 SRB's and 3 SSME's. Each of the three SSME's weighs 3,039 kg. Altogether the propulsion system of the space shuttle weighs 1,918,571 kg. The space shuttle at liftoff weighs about 2,041,166 kg with a maximum cargo capacity of 28,803 kg. After 8.5 minutes of rocket acceleration, the space shuttle reaches an altitude of 45 km, the exhausted ET is jettisoned and the 2 SRB's also separate, leaving the shuttle's 3 main engines the task of boosting the orbiter from 4,828 km/h to 27,358 km/h in 6 minutes to reach orbit. Without the ET, the SSME's do not have much fuel to go far. Thus, it takes the space shuttle a total of 14.5 minutes after launch and a consumption of 1,707,345 kg of propellant just to carry 28,803 kg of payload into orbit 354 km (220 miles) above Earth. As Table 2-3 shows, even this astounding velocity undergoes a tiny time dilation factor of 1.0000000003208 since it is only 1/39,541th the speed of light. We are a long way from interstellar travel, indeed.

As a continuation of the graph in Figure 2-8, we now take the data from Table 2-3 and start at a very high speed v (in terms of c), or which amounts to the same thing, $\beta = v / c$. We graph the times corresponding to $v = 0.9\ c$ to $v = 0.999\ 999\ 999\ 999\ 999\ c$ (15 decimal places), coming as unrealistically close to c as we can get within the limits of floating point numbers allowed by Excel. Note that the top of the graph shows that if an astronaut can manage to journey at velocity $v = 0.999\ 999\ 999\ 999\ 999\ c$, every second she travels through space is equivalent to 22,369,621 seconds on Earth!

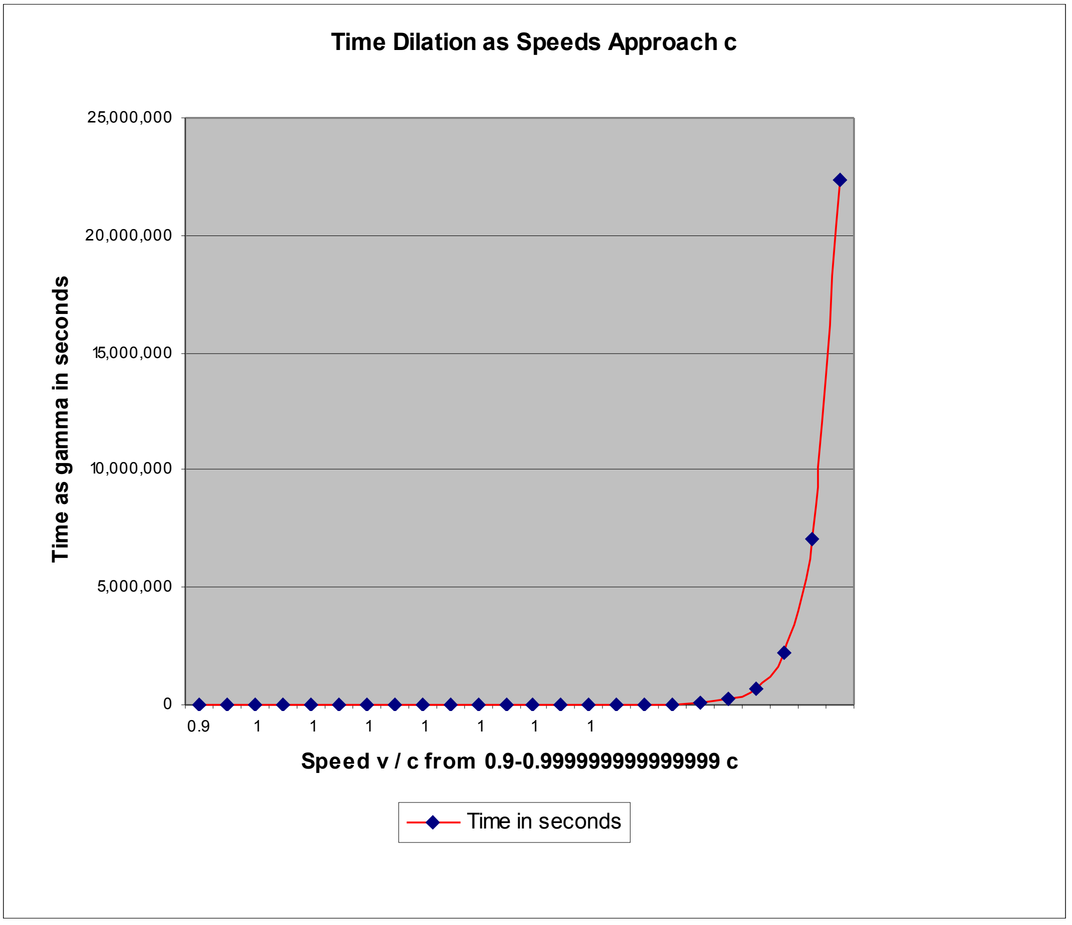


Figure 2-9. Time dilation growing rapidly at very high speeds approaching *c*.

In calculating time dilation, it is important to identify the two events (e.g., the beginning and end of a journey, the beginning and end of a light pulse, the beginning and end of a heartbeat, the creation and decay of a particle, etc.) and the **proper time Δt_0**, which is the time interval between two events that occur at the same location for an observer, e.g., in a spaceship. **The proper time is the shortest time between these two events**. The time interval **Δt** measured by a stationary observer on Earth observing the same two events is not the proper time because the events occur in two different locations.

A consideration of the time dilation formula $\Delta t = \Delta t_0 / (1 - v^2 / c^2)^{1/2}$ shows that the higher the speed v is, the higher γ becomes and the greater Δt becomes. **Δt is greater than Δt_0 by the factor γ at any speed $v > 0$.** However, in $v << c$, the time difference between Δt and Δt_0 is so small it can be detected only by atomic clocks capable to measuring nanoseconds.

We have seen that time runs slower for a traveling observer. This does not mean that the clock runs slower; the clock is accurate and functions normally. Instead, it is in the nature of time to slow down (to dilate) at very high speed compared to time as measured by a stationary observer. Time in an environment moving with high speed with respect to Earth marches to a different drumbeat than time on Earth. We say, again, that **time is relative, i.e., relative to the reference frame**. The obvious question is, What about the normal physical and biological processes that occur in a human traveling at high speed? The answer is that every physical process from heartbeat, digestion, metabolism to breathing and aging slows down accordingly, by exactly the same dilation factor γ. The upshot is that everything from the ticks of the clock to a human's life processes proceed as they would normally do on Earth. The traveling astronauts would notice nothing different in themselves or in their clocks.

To sum up, both the high-velocity space traveler and the earthbound observer live their lives normally in their respective environments. The effects of time dilation become measurable when the space traveler comes back to Earth. **The traveler ages less than the observer on Earth**. Space travel, however, leads to a dilemma we will consider in the next section.

2.3.2 The Twin Paradox

As illustrated in Example 18 above, if you travel with the speed of 0.997 *c*, you will make a 40-ly trip to a star and back in only 3.02 years. Father Daedalus made the 40-ly interstellar trip in 3.02 years while his son Icarus, who stayed on Earth, aged 39 years during the same trip, *making him older than his father on the latter's return*, a scenario rife with promises for science fiction writers. This startling possibility is predicted by the special theory of relativity. To the space travelers their ship is at rest, and the stars and everything else move past them. To observers outside the spaceship, it is the spaceship that moves, and everything else is at rest. Whether Daedalus thinks he is at rest and his son moves or vice versa, or whether he thinks he moves and his son is at rest or vice versa makes no difference in the result. Each side is right in his claims. Each point of view (or reference frame) is equivalent to the other. And in accordance with the first postulate of special relativity, each inertial reference frame is treated the same by the laws of physics. There are no preferred reference systems.

If all reference frames are equivalent, can the space traveler Daedalus claim that it was his son Icarus that traveled while he was at rest, and therefore should be the one that aged more slowly? Can Icarus claim that his father aged more slowly because he is the one that traveled? Put differently, if there is complete symmetry between reference frames, are both claims equally valid? The answer is no because the *two reference systems are not equivalent*. Father Daedalus' spaceship had to accelerate many times to reach its cruising speed of 0.997 *c* and had to accelerate again on the return trip. His spaceship is not an inertial system. But Son Icarus was in an inertial frame, that of the Earth.

If we now substitute two identical twins Tweedledee and Tweedledum for Daedalus and Icarus, the situation does not change. By the first postulate of the special theory of relativity, Tweedledee the traveler can claim that he is stationary and it is his brother who travels. So Tweedledum should age more slowly than Tweedledee. Tweedledum, on his part, will protest that it is his brother Tweedledee that travels and therefore should age more slowly. In this case, we have a putative paradox known as the **Twin Paradox.** By the first postulate each twin can claim that it is his brother who ages more slowly. Yet, both twins simply cannot age more slowly than each other at the same time.

Again the resolution resides in the definition of the inertial frame. If we remember that only Tweedledee has to accelerate in his interstellar voyage, the apparent paradox dissolves. This is a clear case involving an inertial system (Tweedledum) and an accelerating system (Tweedledee). The two systems are not equivalent. Hence, the first postulate does not apply and there is no paradox. We will see accelerating systems treated by the general theory in Chapter 3.

2.3.3 Decay of the Muon

Another experimental confirmation of the special theory of relativity involves the behavior of the muon, an unstable subatomic particle created by cosmic radiation high in the earth's atmosphere that travels down to the earth at a speed approaching the speed of light and has an average life of 2.2 μs (2.2 microseconds or millionths of a second) before it decays.

Consider a muon created at an altitude of 5000 m above Earth's surface. If the muon travels down toward Earth at the speed of 0.995 *c*, it will decay after traveling only:

$$d = 0.995 \text{ x } 3 \text{ x } 10^8 \text{ m x } 2.2 \text{ x } 10^{-6} \text{ s} = 657 \text{ m}.$$

Another way to look at this is from the point of view of the muon itself. If an observer could accompany the muon, she would notice that the muon was at rest and the earth was moving up at the speed of 0.995 c traveling 657 m before the muon decayed. Since the muon is created somewhere at the top of the atmosphere and decays somewhere near the bottom of the atmosphere, to the muon this distance is the thickness of the atmosphere. Large numbers of muons have been detected on the ground because as they travel at near the speed of light, they age more slowly due to the effect of time dilation in their descent and thus live long enough to reach the ground.

Time dilation for radioactive particles was first measured in 1941 by Bruno Rossi and D.B. Hall. But the more notable experiment was conducted in 1963 by the physicists D. H. Frisch and J. H. Smith. In the article "Measurement of Relativistic Time Dilation Using μ-Mesons" published in *American Journal of Physics*, May 1963, they described in detail an experiment in which they counted the number of muons near the peak of Mount Washington, New Hampshire, and the number of muons in their laboratory in Cambridge, Massachusetts. Because of the muon's short life, it was expected that there would be more of them surviving at higher altitudes than at sea level. Frisch and Smith set their detector to detect muons traveling with speeds v within the narrow range of 0.995 c and count the number of muons as they came down near the summit of Mount Washington before their decay. In one experiment run their apparatus found a mean muon life of 2.2 μs, and an average arrival rate of 563 muons per hour The muons' journey through a vertical distance d of 1907 m from their setup on Mount Washington to their laboratory in Cambridge, measured by a clock at rest relative to the laboratory, takes a time:

$$\Delta t = d / v = 1907 \text{ m} / (0.995)(3 \text{ x } 10^8 \text{ s}) = 6.4 \text{ x } 10^{-6} \text{ or } 6.4 \ \mu\text{s}$$

According to their calculations, after the time Δt = 6.4 μs has elapsed, only 27 muons per hour would be expected to survive the trip. However, the detector at Cambridge counted an average hourly survival rate of 408 muons. This rate, according to the experimenters' calculations, corresponds to a mean life of Δt_{0} = 0.7 μs.

To find the velocity with which the muons fell to Earth, given the proper time Δt_{0} = 0.7 μs, i.e., the mean life at rest, and the stationary time Δt = 6.4 μs, we use the time dilation findings by applying the speed formula to obtain:

$$v = (1 - \Delta t_0^2 / \Delta t^2)^{1/2} \ c \qquad (24)$$

$$v = (1 - 0.7^2 \ \mu\text{s} / 6.4^2 \ \mu\text{s})^{1/2} \ c = 0.994 \ c$$

which is incredibly close to the experimental speed of 0.995 c set for the detectors, thereby confirming time dilation and the special theory of relativity.

Further particle experiments in 1985 with fast-moving atoms of neon excited by a laser to obtain high-precision frequency also confirmed time dilation.

Let us take a few more numerical examples to illustrate.

Example 24, If a cosmic ray-produced muon's mean life at rest (i.e., traveling with the muon) is 2.2 μs, (a) how far does the muon travel at this speed of decay? (b) what is its mean life when it travels with the speed of 0.99 c? (c) How far does it travel at this speed before decaying?

Solution. (a) In the muon's frame of reference, traveling for 2.2 μs, the muon covers a distance d_{muon} of:

$$d_{muon} = 0.99 \text{x } 3 \text{ x } 10^8 \text{ m x } 2.2 \text{ x } 10^{-6} \text{ s} = \underline{653 \text{ m.}}$$

From the muon's frame, the earth's atmosphere is only 653 m thick.

(b) The two events, the birth and decay of the muon, occur in the same location only in a rest frame. Given the muon's mean life at rest is the proper time Δt_0, we use the time dilation formula (17) to find the time Δt:

$$\Delta t = \Delta t_0 \ / \ (1 - v^2 / c^2)^{1/2} \qquad (17)$$

$$\Delta t = 2.2 \ \mu\text{s} \ / \ (1 - 0.99^2 \ c^2 / c^2)^{1/2} = \underline{15.6 \ \mu s}.$$

(c) According to an observer on Earth, the distance d_{Earth} traveled by the muon during this time is:

$$d_{Earth} = \Delta t\ v = (15.6 \times 10^{-6}\ \text{s})\ (0.99)\ (3 \times 10^{8}\ \text{m/s}) = \underline{4632\ m.}$$

For the earthbound observer, the muon has traveled 4632 m to arrive close to the ground level.

Example 25, Assuming a muon is created at an altitude of 5500 m and has a mean life at rest of 2.2 μs, (a) what is its mean life when it travels with the speed of 0.95 *c*? (b) what is the distance it travels to Earth before decaying?

Solution. (a) Since the muon's mean life at rest is its proper time Δt_0, we use the time dilation formula (17) to find its mean life when traveling with velocity 0.95 *c*.

$$\Delta t = \Delta t_0 \ /\ (1 - v^2 / c^2)^{1/2} \qquad (17)$$

$$\Delta t = 2.2\ \mu\text{s} \ /\ (1 - 0.95^2\ c^2 / c^2)^{1/2} = \underline{7.05\ \mu s \text{ or } 7.05 \times 10^{-6}\ s.}$$

(b) The distance the muon travels at the speed $v = 0.95 \times 3 \times 10^8$ m/s $= 2.85 \times 10^8$ m/s:

$$\mathbf{d = \Delta t\ v}$$

$$d = 7.05 \times 10^{-6}\ \text{s} \times 2.85 \times 10^{8}\ \text{m/s} = \underline{2009\ m.}$$

Example 26, What is the mean life of a muon measured in the laboratory frame of reference if its mean life at rest is 2.20×10^{-6} s and its speed is 2.00×10^8 m/s relative to the lab?

Solution. The muon's mean life at rest is its proper time Δt_0. Applying the time dilation formula yields:

$$\Delta t = \Delta t_0 \ /\ (1 - v^2 / c^2)^{1/2} \qquad (17)$$

$$\Delta t = 2.20 \times 10^{-6}\ \text{s} \ /\ [1 - (2.00 \times 10^{8}\ \text{m/s})^2 / (3.00 \times 10^{8}\ \text{m/s})^2]^{1/2}$$

$$\Delta t = 2.20 \times 10^{-6}\ \text{s} \ /\ [1 - (2/3)^2]^{1/2} = 2.2 \times 10^{-6}\ \text{s} / [1 - (4/9)]^{1/2} = \underline{2.95 \times 10^{-6}\ s.}$$

Example 27, A muon's average lifetime at rest is found to be 2.5×10^{-6} s, and the time it takes to reach the laboratory is 6.8×10^{-6} s. Find (a) the speed with which it traveled, and (b) the distance it traveled.

Solution. (a) Here we have $\Delta t_0 = 2.5 \times 10^{-6}$ s and $\Delta t = 6.8 \times 10^{-6}$ s. Applying the speed equation, we obtain:

$$v = (1 - \Delta t_0^2 \ /\ \Delta t^2)^{1/2}\ c \qquad (24)$$

$$v = [1 - (2.5 \times 10^{-6}\ \text{s})^2 \ /\ (6.8 \times 10^{-6}\ \text{s})^2]^{1/2}\ c = \underline{0.93\ c.}$$

(b) The distance the muon travels at the speed $v = 0.93\ c$:

$$d = \Delta t\ v$$

$$d = 6.8 \times 10^{-6}\ \text{s} \times 0.93 \times 3 \times 10^{8}\ \text{m/s} = 1.897 \times 10^{3}\ \text{m} = \underline{1897\ m.}$$

Example 28, The Conseil européen pour la recherche nucléaire (CERN, European Council for Nuclear Research), straddling France and Switzerland outside Geneva, is the premier site of particle physics research center equipped with the world's largest particle accelerators and detectors. It produced muons accelerated to high speeds with a mean life 30 times greater than the muon's mean lifetime at rest. What is the speed of a CERN muon?

Solution. The mean life of a muon has been determined to be $\Delta t_0 = 2.2 \times 10^{-6}$ s. The CERN muon's mean life is $\Delta t = 2.2 \times 10^{-6}$ s x 30 = 66.0×10^{-6} s. Applying the speed formula yields:

$$v = (1 - \Delta t_0^2 / \Delta t^2)^{1/2}\ c \tag{24}$$

$$v = [1 - (2.2 \times 10^{-6}\ \text{s})^2 / (66.0 \times 10^{-6}\ \text{s})^2]^{1/2}\ c = \underline{0.9994\ c.}$$

Example 29, Produced by cosmic rays, a muon travels 3000 m to Earth with a mean life in its fall of 44 x 10^{-6} s. What are its speed *v,* its γ, and its proper time?

Solution. (a) Knowing the muon's mean life in fall and the distance traveled, we use the distance formula to derive its velocity:

$$d = \Delta t\ v$$

$$v = d / \Delta t$$

$$v = 3000\ \text{m} / 44 \times 10^{-6}\ \text{s} = \underline{66.18 \times 10^6\ \text{m/s or } 0.66\ c.}$$

(b) The factor γ is derived from:

$$\gamma = (1 - v^2 / c^2)^{-1/2} \tag{3}$$

$$\gamma = (1 - 0.66^2\ c^2 / c^2)^{-1/2} = \underline{1.33.}$$

(c) The proper time is derived from:

$$\Delta t_0 = \Delta t\ (1 - v^2 / c^2)^{1/2} \tag{21}$$

$$\Delta t_0 = 44 \times 10^{-6}\ \text{s}\ (1 - 0.66^2\ c^2 / c^2)^{1/2} = \underline{33 \times 10^{-6}\ \text{s.}}$$

2.4 Length Contraction

Just as time and simultaneity are relative to reference frames, distance or length is also relative to reference frames, as we will see shortly. If someone tells you that a starship flying at close to the speed of light (reference frame *S'*) *shrinks in length* in the direction of motion with respect to an observer on Earth (reference frame *S*), what will you think? Not only does the starship shrinks, everything else in it, including humans, also shrinks as measured in the *S* frame. And if the starship manages to approach the speed of light, its length shrinks to zero! You will probably feel very uncomfortable with the notion of a moving object's shrinking simply by traveling with a speed close to *c*. Yet, this is what nature is like and the phenomenon conforms to the special theory of relativity.

2.4.1 Length Contraction

The shortening of distances must by now be familiar to the reader, given what we have seen in the Lorentz-FitzGerald contraction as well as in time dilation.

We saw in Section 2.1.4 how scientists were trying to explain the null result of the Michelson-Morley experiment by positing an ad-hoc factor known as the Lorentz-FitzGerald contraction, which turns out to be the factor γ.

Recall Example 18 above, in which an astronaut father, Daedalus, came home younger than his earthbound son Icarus after a voyage to a star 20 ly away by a measurement made from Earth. Daedalus covered the distance of 40 ly in just $\Delta t_0 = 3.02$ years traveling at the speed of 0.997 *c* whereas to his son the same trip took $\Delta t = 39$ years by a clock on Earth. This distance *d* of 3.02 y x 0.997 *c* amounts to only 3.01 ly as compared to the 40 ly measured from the Earth's point of view. Clearly, distance has shrunk for the space traveler. This shortening of distance coupled with the slowing down of time makes space travel a very attractive proposition. The one major hurdle, aside from financial matters, is the technology of propulsion powerful enough to reach very high speeds.

Now take Example 24 about the muon produced in the upper atmosphere by cosmic rays. From the muon's reference frame, it travels only 653 m while from the Earth's frame, it travels 4632 m at velocity 0.99 *c*. Clearly, distance contracts for the muon.

When the concept of absolute simultaneity was overturned, it brought down with it the concept of absolute time as well as that of absolute length or distance. To measure the length of an object, e.g., the length of a meter stick, we need to see both ends of it at the same time. But since two events that are simultaneous in one frame of reference *S'* are not necessarily simultaneous in another reference frame *S* in motion with respect to the first, we are faced with a measurement problem. The measurements of time and distance in these frames will differ.

To examine this issue quantitatively, let us consider a thought experiment. A spaceship travels from star A to star B with a speed v. An observer from Earth measures the distance between the stars as L_0 (called the **proper length**), and according to this observer, the trip takes the time:

$$\Delta t = L_0 / v \quad \text{or} \quad v = L_0 / \Delta t$$

For their part the spaceship astronauts maintain that they are at rest and the destination star is moving toward them at the speed v. According to the astronauts the distance in the direction of motion between the stars is L, and the time it takes them to reach their destination is:

$$\Delta t_0 = L / v \quad \text{or} \quad v = L / \Delta t_0$$

Since observer and astronauts agree that the spacecraft's speed is v in either case, the above relations can be rewritten to eliminate v:

$$L / \Delta t_0 = L_0 / \Delta t$$

$$L = L_0 \, \Delta t_0 / \Delta t$$

But formula (23) has established the relation between Δt_0 and Δt as γ:

$$\gamma = \Delta t / \Delta t_0 \qquad (23)$$

Hence, the formula for relativistic length contraction in the direction of motion is:

$$\boldsymbol{L = L_0 \gamma^{-1} \text{ or } L = L_0 (1 - v^2 / c^2)^{1/2}} \qquad \textbf{(25)}$$

And the spreadsheet equivalent of the length contraction formula is:

$$\mathbf{L = L_0 * (1 - (v^\wedge 2 / c^\wedge 2))^\wedge(1/2)} \qquad \textbf{(26)}$$

Substitute a value for the proper length $\boldsymbol{L_0}$ and the speed $\boldsymbol{v}$ to obtain the contracted length $\boldsymbol{L}$**.**

Given a contracted length L and the speed v, **we derive the proper length** $\boldsymbol{L_0}$ as follows:

$$\boldsymbol{L_0 = \gamma L \text{ or } L_0 = L / (1 - v^2 / c^2)^{1/2}} \qquad \textbf{(27)}$$

which, when translated into a spreadsheet formula, becomes:

$$\boldsymbol{L_0 = L / (1 - (v^\wedge 2 / c^\wedge 2))^\wedge(1/2)} \qquad \textbf{(28)}$$

If v is expressed in terms of c, use the following formula:

$$\boldsymbol{L_0 = L / (1 - (v^\wedge 2))^\wedge(1/2)} \qquad \textbf{(29)}$$

Substitute the contracted length L and v in terms of c in the above formula to obtain the proper length.

Solving for the speed v, we get:

$$L = L_0 (1 - v^2 / c^2)^{1/2}$$

$$L^2 = L_0^2 (1 - v^2 / c^2)$$

$$L^2 / L_0^2 = 1 - v^2 / c^2$$

$$v^2 / c^2 = 1 - L^2 / L_0^2$$

$$v^2 = (1 - L^2 / L_0^2)\ c^2$$

$$\mathbf{v = c\,(1 - L^2 / L_0^2)^{1/2}} \qquad \mathbf{(30)}$$

or, if we want to find the speed v in terms of c:

$$\mathbf{v / c = (1 - L^2 / L_0^2)^{1/2}} \qquad \mathbf{(31)}$$

Translating relation (30) into a spreadsheet formula, we get the speed *v*:

v = c*(1 – L^2 / L₀^2)^(1/2) **(32)**

or **the speed *v* in terms of** c:

v = (1 – L^2 / L₀^2)^(1/2) **(33)**

Since the reciprocal of $\gamma < 1$, $L < L_0$. The proper length is greater than the length as measured by the traveling astronauts by the reciprocal of the factor γ. This phenomenon is called **length contraction.**

The distance L_0, called the *proper length*, is the length of an object measured by an observer at rest with respect to that object. In our example, the proper length is the distance between star A and star B as measured by an earthbound observer. The distance L is measured by the traveling astronauts. The proper length L_0 is greater than the contracted length L of the same object in motion. And the length contracted is shorter than the proper length by the factor γ^{-1}.

Notes: (1) The astronauts travel during the proper time interval Δt_0 but cover the distance L,. *not the proper length* L_0. On the other hand, the earthbound observer measures the distance between the stars as the *proper length* L_0, but measures the time the astronauts take to cover this distance as the stationary time Δt. Therefore, *it should never be assumed that an observer that measures the proper time also measures the proper length or vice versa.*

(2) Length contraction occurs only in the length that is parallel to the direction of motion of the object. It does not occur in the dimension perpendicular to the direction of motion, i.e., the object's height remains unchanged, assuming the motion is horizontal.

(3) The factor γ approaches 1 as the speed v approaches everyday speeds, $v << c$, and length contraction becomes unnoticeable. When $v = 0$, i.e., when the object is at rest, $\gamma = 1$, and there is no length contraction.

(4) Given that no object travels faster than light, i.e., $v > 0$, no object has length zero in any reference frame. See the graph in Figure 2-11 below.

To the question "How long is a meter stick?", the answer is, "It depends on whether the stick moves and how fast it moves, and whether the observer who measures it moves or not and at what speed." Quantitatively, length measurements depend on the factor γ, which in turn depends on the speed v of the system. We conclude that *there is no such thing as absolute length (distance or space).* **Length, like motion, time and simultaneity, is relative**.

Effects of Length Contraction

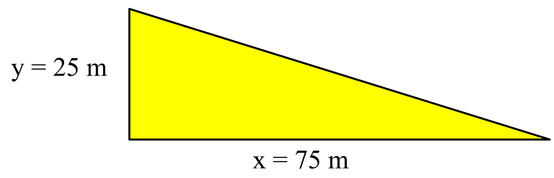
Length contraction has startling effects. A spaceship that passes you at near the speed of light not only appears shorter but also shows its rear surface even though you are directly alongside it. This effect, which is not a relativistic effect, occurs because of the finite speed of light. By the time light from the side of the spaceship reaches you the spaceship has moved. Light from the rear of the spaceship must have left earlier than light from the side in order to reach you at the same time. By the same token, if you fly past a building at nearly the speed of light, you can see both the front and side of a building as you pass it. This effect makes it wrong for science-fiction writers to represent buildings or people to be skinny and elongated in the vertical direction.

Length contraction allows the muon created by cosmic radiation high up in the atmosphere to reach ground level in spite of its short life.

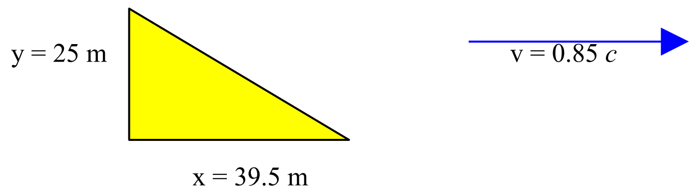
Take the example of a triangular spaceship in Figure 2-10 that moves to the right with a speed of $v = 0.85\ c$. Its length x at rest (i.e., its proper length) measures 75 m and its height y at rest is 25 m. At the speed v, the ship's contracted length is given by formula (25):

$$L = L_0 (1 - v^2 / c^2)^{1/2} \qquad (25)$$

$$L = 75 (1 - 0.85^2)^{1/2} = 39.5 \text{ m}$$



(a) Spaceship at rest.



(b) Spaceship moving with v = 0.85 *c*.

Figure 2-10. (a) Length contraction occurs only in the direction of motion, affecting the spaceship's length x. (b) Length contraction does not occur in the direction perpendicular to the direction of motion. Therefore, the ship's height y remains unchanged.

Quantitative examples of length contraction follow.

Example 30, A spaceship measuring 90 m on the runway travels at a speed of 0.65 c with respect to an earthbound observer. How long does the spaceship in flight appear to the same observer?

Solution. Given the proper length L_0 and the speed v, we find the contracted length L of the spaceship with:

$$L = L_0 (1 - v^2 / c^2)^{1/2} \qquad (25)$$

$$L = 90 (1 - 0.65^2)^{1/2} = \underline{68.39 \text{ m.}}$$

Example 31, A spaceship travels past the earth at a speed of 0.75 c relative to an earthbound observer. How long does a meter stick in the spaceship appear to the same observer?

Solution. Given the proper length L_0 and the speed v, we find the contracted length L of the meter stick with:

$$L = L_0 (1 - v^2 / c^2)^{1/2} \quad (25)$$

$$L = 1 (1 - 0.75^2)^{1/2} = \underline{0.66 \text{ m.}}$$

Example 32, To an observer on Earth the 100m-long rocket ship measures 55.85 m. How fast does the rocket ship fly past the observer?

Solution. We have the rocket ship's proper length L_0 and its contracted length L. Its speed is given by the formula

$$v / c = (1 - L^2 / L_0^2)^{1/2} \quad (31)$$

$$v / c = (1 - 100^2 \text{ m} / 55.85^2 \text{ m})^{1/2} = \underline{0.83\ c.}$$

Example 33, If a spaceship as measured by a stationary observer is 37.2 m long as it moves past the observer at a speed of 0.65 c, how long is it when at rest on the ground?

Solution. Given the contracted length L and the speed v, we find the proper length L_0 of the rocket ship with:

$$L_0 = L / (1 - v^2 / c^2)^{1/2} \quad (27)$$

$$L_0 = 32.7 \text{ m} / (1 - 0.65^2)^{1/2} = \underline{43.0 \text{ m.}}$$

Example 34, At what speed does a spaceship have to travel past the earth for an earthbound observer to measure its length to be 150 m if it is 175 m long when at rest?

Solution. Given the contracted length L and the proper length L_0 of the spaceship, we derive its speed with:

$$v / c = (1 - L^2 / L_0^2)^{1/2} \quad (31)$$

$$v / c = (1 - 150^2 / 175^2)^{1/2} = \underline{0.52\ c.}$$

Example 35, A rocket ship hurtles through space at the speed of 0.35 c to an earth observer. (a) Find its γ. (b) Find its contracted length L as observed by the earth observer. (c) Find its ΔH, the change in its height in the direction perpendicular to its motion, as observed by the earth observer. (d) Find its contracted length L as observed by an onboard astronaut. (e) Calculate the change ΔL in its length in the direction of motion as observed by the earth observer.

Solution. (a) The contraction factor γ for speed $v = 0.35\ c$ is given by:

$$\gamma = 1 / (1 - \mathrm{v}^2 / \mathrm{c}^2)^{1/2} \quad (3)$$

$$\gamma = 1 / (1 - 0.35^2\ \mathrm{c}^2 / \mathrm{c}^2)^{1/2} = \underline{1.07.}$$

(b) The rocket ship's contracted length L as observed by the earth observer is given by:

$$L = L_0 (1 - v^2 / c^2)^{1/2} \quad (25)$$

$$L = L_0 (1 - 35^2\ c^2 / c^2)^{1/2} = \underline{0.94\ L_0}.$$

(c) Since height is perpendicular to the direction of motion, there is no change in the rocket ship's height.

(d) Since the onboard astronaut moves with the rocket ship, there is no change in the ship's length observed by the astronaut.

(e) The change ΔL is measured by comparing L and L_0:

$$\Delta L = L - L_0\,(1/\gamma - 1) = 0.94\,L_0 - (0.9367 - 1)\,L_0 = \underline{-0.0594\,L_0.}$$

Example 36. How long does an earthbound observer measure a 150-m-long spaceship to be when it flies past the observer (a) with a velocity of 0.64 c? (b) with a velocity of 0.001 c?

Solution. Given the proper length L_0 of the spaceship and its speed, we derive its the contracted length L:

(a) at speed 0.64 c:

$$L = L_0\,(1 - v^2/c^2)^{1/2} \tag{25}$$

$$L = 150\text{ m}\,(1 - 0.64^2\,c^2/c^2)^{1/2} = \underline{115\text{ m.}}$$

(b) at speed 0.001 c:

$$L = L_0\,(1 - v^2/c^2)^{1/2} \tag{25}$$

$$L = (150\text{ m})\,(1 - 0.001^2\,c^2/c^2)^{1/2} = \underline{149.9999\text{ m.}}$$

Example 37. From an altitude of 550 m a spaceship moves downward toward Earth with a velocity of 0.99 c. What is the altitude of the spaceship as measured by an astronaut in the spaceship?

Solution. Note that the direction of motion is vertically downward, and therefore length contraction occurs. Given the proper length L_0, which is the spaceship's altitude, and its speed, we derive the altitude (contracted length) L as measured by the astronaut:

$$L = L_0\,(1 - v^2/c^2)^{1/2} \tag{25}$$

$$L = (550\text{ m})\,(1 - 0.99^2\,c^2/c^2)^{1/2} = \underline{77.59\text{ m.}}$$

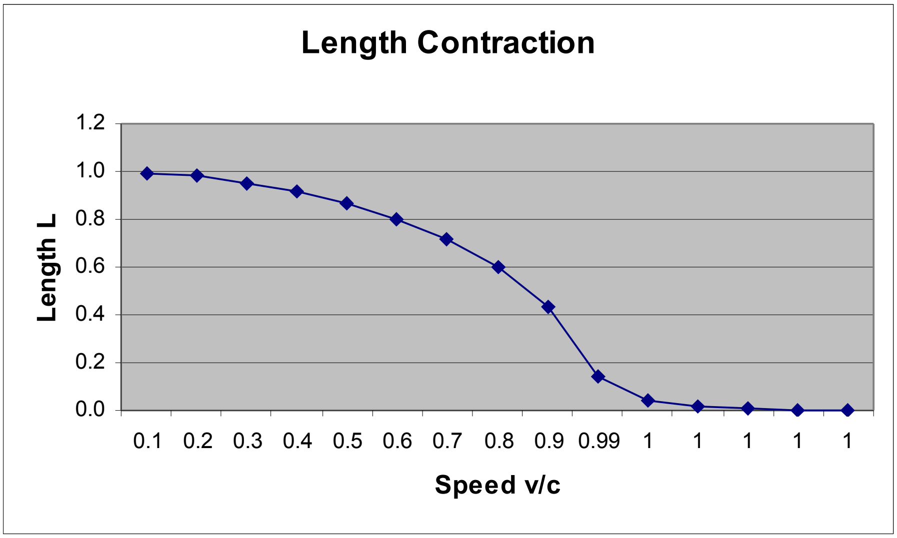


Figure 2-11. Contraction of a meter stick of length *L* as it moves at speed *v / c*. As its speed approaches the speed of light, its length approaches zero. In principle, at $v = c$, $L = 0$. However, since nothing moves as fast as light, $v > c$, and $L > 0$. Each of the x coordinates after 0.99 has an additional 9 in the next decimal place.

Figure 2-11 shows how the length of a moving object (or distance) appears contracted to an observer at rest. Contraction increases with speed and becomes extreme (approaches zero) at very high speeds. But, given that no object can travel at the speed of light, no distance or length is reduced to zero. This is another sense in which *c* is considered the ultimate speed.

Note that objects only appear contracted from the stationary observer's reference frame. From the reference frame of the moving object, however, nothing in that object contracts. A meter stick on a space ship traveling at 0.9 *c* measures 1 meter just as it does when the ship is at rest on the ground. By reason of symmetry the moving observer finds the stationary stick to be shortened. More importantly, length contraction is related to time dilation. Recall in Section 2.3.3 the muon's journey to Earth's surface from its atmospheric height of 4000-5000 m. From its point of view, the distance it travels is short (we call it contracted) enough to cover before it decays in nanoseconds. Though the muon thinks it travels only about 600 m in its brief life, it actually arrives near the ground. And the muons reach the ground in large enough quantity to be detected. Also, to a stationary observer, a diner on a spaceship traveling at a high speed appear to take more time, e.g., 20 minutes, to finish a meal that is served on a plate whose diameter has shrunk to 15 cm. To the traveling diner, the meal takes only 15 minutes and his plate measures 20 cm in diameter.

Visual Effects of Length Contraction

For some 50 years after the special theory had been proposed, it was thought that length contraction could be observed or photographed directly. For instance, an astronaut moving at high speed in a spaceship could take a photograph of another identical ship moving in the opposite direction that shows its shortened length in the direction of motion. However, there is a difference between *observing* length contraction and *seeing* length contraction. In fact, Roger Penrose in 1958 and James Terrell in 1959, working independently, found that the Lorentz contraction of distant objects is more of a rotation than a shortening.

The following abstract of the article "Invisibility of the Lorentz Contraction" by the American physicist James Terrell, published in the journal *Physical Review* (116, pp.1041–1045, 1959), provides an overview:

> It is shown that, if the apparent directions of objects are plotted as points on a sphere surrounding the observer, the Lorentz transformation corresponds to a conformal transformation on the surface of this sphere. Thus, for sufficiently small subtended solid angle, an object will appear—optically—the same shape to all observers. A sphere will photograph with precisely the same circular outline whether stationary or in motion with respect to the camera. An object of less symmetry than a sphere, such as a meter stick, will appear, when in rapid motion with respect to an observer, to have undergone rotation, not contraction. The extent of this rotation is given by the aberration angle (*theta* – *theta*'), in which *theta* is the angle at which the object is seen by the observer and *theta'* is the angle at which the object would be seen by another observer at the same point stationary with respect to the object. Observers photographing the meter stick simultaneously from the same position will obtain precisely the same picture, except for a change in scale given by the Doppler shift ratio, irrespective of their velocity relative to the meter stick. Even if methods of measuring distance, such as stereoscopic photography, are used, the Lorentz contraction will not be visible, although correction for the finite velocity of light will reveal it to be present.

This effect is known as the **Terrell effect**. In order to see the relativistic effects of length contraction, we need to slow down the speed of light *c* and travel with it. This can be accomplished by computer simulation. In the article "Seeing relativity: Visualising special relativity" (1997-1999), Anthony Searle, of the University of New South Wales, Australia, described the visual effects in videos made from computer simulation of *reduced c scenes.* Making light travel 5 m/s, we will see the effects of slow light. First, we observe the effects on a stationary frame. A streetlamp does not instantly flood the scene with light but emits spherical balls of light that gradually expand in all directions until they hit a surface and make it visible to us. If we flip the lamp on and off, we see the lamp turn on first, then parts of the ground and finally the wall even though the wall is closer to the streetlamp than the ground. This is because light takes less time to reach our eyes via the ground (the shortest distance) than via the wall (a longer distance). If we now consider a moving frame, such as a tram moving at 0.866 *c*, which is gamma = 2, the effects are different. The tram looks distorted, its colors change, and their intensity also change. Even its shadow slants at an unusual angle. We can study each effect separately: distortion as angular aberration, color change as the Doppler effect, and brightness as the headlight effect by eliminating the other effects.

If we study distortion, the tram not only appears shorter but also sheared and slightly bent. Shear is the effect that results in the tram's sides deviating from the perpendicular. First as the tram moves at near-light speed, it undergoes the Lorentz contraction. Then we see the back side even before we see the front surface, resulting in the shearing effect. This means light coming from the back, which has a longer distance to travel to our eye, must have left earlier than light coming from the front. What we are seeing is the image of the back side at an earlier time. We call this combination of contraction and shear the **Terrell effect** or the **Penrose-Terrell rotation**. This effect occurs with objects that are small and distant. At close quarters, however, different parts of the tram appear rotated by different amounts, resulting in extreme distortion. Lines perpendicular to an observer's direction of motion will appear curved while lines parallel will appear contracted. The whole tram (or a cube) will appear rotated. The observer's field of view appears compressed in the forward direction, and expanded in the reverse direction. Also objects behind the observer will be rotated into sight. Note that the Penrose-Terrell rotation is only an optical effect, for what really rotates is the image and not the tram. If the object is a sphere, its length contraction is exactly offset by its rotation so that it retains its shape to an observer.

2.5 Relativistic Momentum

We have seen that in relativistic terms, there is no absolute simultaneity, velocity, time, or length. They are all relative to the frame of reference in which they are measured. We now examine one of the most important quantities of mechanics, which is the vector called **momentum**, and determine whether momentum behaves in the same way classically and relativistically. In Newtonian mechanics, the **linear** momentum of an object equals the product of its mass and velocity, as denoted by:

$$p = mv \qquad (34)$$

where p is the momentum of an object, m its mass, and v its velocity, expressed in the SI unit kg.m/s^2. Note that this relation does not set the upper limit of the velocity.

A fundamental law in Newtonian mechanics, the **conservation of momentum** states that *when two objects (e.g., balls, cars, trains, and so on) collide, the total momentum of the system remains constant before and after the collision, although the momentum of each object may change, provided no external forces come into play*. In other words, momentum is conserved in **isolated systems**, where the only forces exerted are internal to the objects, which may range from particles to galaxies or their individual stars. We note that the notion of collision between objects must be generalized to all kinds of "collision" whether there is contact between them or not. For example, a proton and an alpha particle, both of which have positive charge and repel each other, still obey the law of **conservation of (linear) momentum**, $\Sigma p = \Sigma p'$, whose general expression is:

$$m_1v_1 + m_2v_2 = m_1v'_1 + m_2v'_2 \qquad (35)$$

where the subscripts refer to the two objects in collision, the unprimed quantities represent the values before the collision, and the primed quantities the values after the collision. The law holds for collisions of all kinds of objects, from hard bodies, soft bodies to subatomic particles and galaxies.

Conservation of momentum must be distinguished from conservation of kinetic energy or energy associated with motion. Whereas momentum is always conserved, **kinetic energy** (KE = ½ mv^2), may or may not. If energy is conserved as in colliding billiard balls, we have an **elastic collision**. After the collision the two bodies move off in different directions. If after the collision, the two bodies stick together, as may sometimes happen in a train collision, we have a **perfectly inelastic collision**. Most of the collisions are **inelastic collisions** as they are neither elastic nor perfectly inelastic.

Consider a one-dimensional head-on collision of two cars in which they stick together. Such a collision is perfectly inelastic and their final velocity v' is therefore the same: $v' = v'_1 = v'_2$, as they now form a system. From the relation (35) we determine the final velocity in terms of the masses and the initial velocities, i.e., using momentum. We get the conservation of momentum in a perfectly inelastic collision as follows:

$$m_1v_1 + m_2v_2 = m_1v'_1 + m_2v'_2 \qquad (35)$$

$$\boldsymbol{m_1v_1 + m_2v_2 = (m_1 + m_2)\, v'} \qquad \textbf{(36)}$$

Hence, the final velocity v' of bodies involved in a perfectly inelastic collision is:

$$v' = (m_1v_1 + m_2v_2) / (m_1 + m_2)$$

$$\mathbf{v' = (m_1v_1) / (m_1 + m_2) + (m_2v_2) / (m_1 + m_2)} \quad \textbf{(37)}$$

Example 38: After a head-on collision between a full-sized car with a mass 1700 kg and an initial velocity of 9 km/h and a small car with a mass of 850 kg and a speed of 12 km/h moving in the opposite direction, they stick together. (a) What is the final velocity of the two cars immediately after the collision? (b) What is the acceleration of each car?

Solution: (a) This is an example of a perfectly inelastic collision, in which the two bodies stick together and move as one. The final velocity v' of the system is:

$$v' = (m_1v_1) / (m_1 + m_2) + (m_2v_2) / (m_1 + m_2) \quad (37)$$

Let us first note that the mass of the small car is only half that of the large one. Hence, $m_1 = 2m_2$, and

$$v' = (2m_2 \text{ x } 9 \text{ km/h}) / (2m_2 + m_2) + (m_2 \text{ x} - 12 \text{ km/h}) / (2m_2 + m_2)$$

Eliminating all m_2 and the common denominator, and noting that the small car moves in the negative direction with respect to the big car, we get the final velocity:

$$v' = 2 (9 \text{ km/h}) - 12 \text{ km/h} = + \underline{6 \text{ km/h.}}$$

The positive sign indicates that the system is moving in the direction of motion of the big car.

(b) The change in speed for the big car is:

$\Delta v_1 = + 6 \text{ km/h} - 9 \text{ km/h} = - \underline{3 \text{ km/h.}}$

And the change in speed for the small car is given by:

$\Delta v_2 = + 6 \text{ km/h} - (- 12 \text{ km/h}) = + \underline{18 \text{ km/h.}}$

Given that **acceleration** is defined as $\Delta v / \Delta t$, and that Δt is identical for both cars, it is clear that the big car has acquired an acceleration equal to only one third the acceleration of the small car. Passengers in the small car, who experience an acceleration three times that of the big car, are subjected to greater forces and greater potential for injuries.

Consider now a case of elastic collision. If a ball dropped from height h rebounds to the same height h', that is $h = h'$, we have an elastic collision; in which case both the kinetic energy and the momentum are conserved. If $h \neq h'$, we have an inelastic collision, in which the momentum is conserved but the kinetic energy is not. The loss of kinetic energy is converted into other forms, such as thermal energy.

Using an elastic collision between two identical particles (or balls), we can investigate the effect of relativity on momentum. Imagine two railroad flatcars S and S' running toward each other on parallel tracks. As they approach each other, ball A is thrown with speed v from S toward S' perpendicular to its motion. At the same time an identical ball B is launched with the same speed v from S' toward S also perpendicular to its motion. Having traveled the same distance y, the two balls collide elastically and rebound before being caught by their respective observers. Ball A with mass m has thus traveled $2y$ in S frame, and ball B with identical mass m has traveled $2y'$ in S' frame. Since there is no length contraction in the y direction, the distances traveled by the balls are the same, $y = y'$. Thus symmetry exists between the two frames of reference.

However, if the situation is analyzed from the point of view of one of the moving frames, e.g., S', time dilation takes effect. As viewed from S', the two events of throwing and catching the ball occur in the same frame and is measured by the *same* clock. They occur in the *proper time* and no time dilation effect emerges. But in S these two events occur in two different locations, and the time measured in S is related to the proper time by the time dilation formula, $\Delta t = \Delta t_0 / (1 - v^2 / c^2)^{1/2}$. Hence, although the balls travel the same distance $y = y'$, the times are different, and the y components of their speeds must also be different.

The velocity of A in S is given by:

$v_A = 2y / \Delta t_0 = v$

whereas the velocity of B in *S'* is:

$v_B = 2y / \Delta t_0 \ (1 - v^2 / c^2)^{1/2} = v(1 - v^2 / c^2)^{1/2}.$

Since the velocities of A and B are not the same, their momenta must differ. Yet, conservation of momentum is so fundamental in classical and relativistic mechanics it must be preserved. After all, by the first postulate of special relativity all laws of physics are the same in all inertial frames. Consequently, momentum must be redefined to take into account the effect of relativity. Since the *y* components of the velocities of A and B differ by the factor $\gamma = (1 - v^2 / c^2)^{1/2}$, we can generalize the relativistic momentum definition as follows:

Relativistic momentum p:

$$\boldsymbol{p = mv / (1 - v^2 / c^2)^{1/2} \quad \text{or} \quad \gamma mv} \qquad \textbf{(38)}$$

SI unit: kg.m/s

As has been observed in other relativistic effects, the factor γ indicates that at everyday speed, $v << c$ (γ close to 1), relativistic momentum reduces to classical momentum. Also the momentum *p* increases as the velocity *v* increases, but *v* never reaches *c*. As the momentum increases, an ever larger force is needed to accelerate the particle further. An infinite amount of energy would be required to achieve the speed of light, making it the upper limit of all speeds.

Example 39: What is the momentum of an electron that travels at 0.9999999997 *c* in terms of *mc*?

Solution: We apply the formula for relativistic momentum.

$$p = mv / (1 - v^2 / c^2)^{1/2} \qquad (38)$$

$$= (m)\, 0.9999999997\, c / (1 - 0.9999999997^2\, c^2 / c^2) = \underline{4.08 \text{ x } 10^4\, mc.}$$

Example 40: What is the momentum of an electron that travels at 0.99 *c* if its mass is 9.1094 x 10^{-31} kg?

Solution: We apply the formula for relativistic momentum.

$$p = \gamma mv \qquad (38)$$

$$= (7.0888)\,(9.1094 \text{ x } 10^{-31} \text{ kg})\,(0.99\, c)$$

$$= (7.0888)\,(9.1094 \text{ x } 10^{-31} \text{ kg})\,(2.9679 \text{ x } 10^8 \text{ m/s}) = \underline{1.9 \text{ x } 10^{-21} \text{ kg.m/s.}}$$

2.6 Relativistic Mass

One of the on-going controversial issues about special relativity is the interpretation of the concept of **relativistic mass**. In the relativistic momentum formula above, *m* is interpreted as the mass of the object as measured by an observer at rest, i.e., the **rest mass**, m_0. The rest mass is also called **invariant mass.** Thus, we can write the momentum equation as:

$$p = \gamma m_0 v \text{ or } m_0 v / (1 - v^2 / c^2)^{1/2} \qquad (39)$$

But the momentum formula $p = \gamma mv$ is also interpreted to mean that as the velocity *v* increases, the mass *m* increases, i.e., mass is velocity-dependent, as relativistic mass is supposed to be, as shown:

$$p = m_0 v / (1 - v^2 / c^2)^{1/2}$$

$$mv = m_0 v / (1 - v^2 / c^2)^{1/2}$$

Dividing both sides by v, we obtain the expression for the **relativistic mass**:

$$\boldsymbol{m = m_0 \,/\, (1 - v^2 / c^2)^{1/2}} \qquad \textbf{(40)}$$

The formula (40) implies that relativistic mass is related to rest mass by the factor γ and increases with the speed v. This motivates some authors to associate the mass in Newton's momentum $p = mv$ with relativistic mass, making it a general relation for all masses. It must be noted that at $v << c$ mass increase is infinitesimally small and therefore imperceptible. Under this interpretation, as the velocity v approaches c, the (relativistic) mass m tends to infinity, as indicated by relation (40). A constant force applied to the mass of an object moving with ever-increasing speed produces ever-decreasing acceleration as $v \rightarrow c$. In other words, if $v = c$ the denominator in relation (40) becomes zero, and the mass m would be infinite. Also it would take infinite energy to generate the speed $v = c$, an impossible feat. This is another reason to consider light as the ultimate speed.

Example 41: What is the relativistic mass of a ball traveling at 0.70 c if its rest mass is 0.4 kg?

Solution: Applying equation (40), we get:

$$m = m_0 \,/\, (1 - v^2 / c^2)^{1/2} \qquad (40)$$

$$= 0.4 \text{ kg} \,/\, (1 - 0.70^2 / c^2)^{1/2} = \underline{0.56 \text{ kg.}}$$

Many beginning physics textbooks as well as popular books on special relativity, including works by Stephen Hawking (*A brief history of time*), Richard Feynman (*Lectures on physics*), and Brian Greene (*The elegant universe*), introduce relativistic mass increase, at least as a heuristic. Michael Fowler (1996), too, defends relativistic mass increase thus:

> Deciding that masses of objects must depend on speed like this ($M = m/(1-v^2/c^2)^{1/2}$, addition mine) seems a heavy price to pay to rescue conservation of momentum! However, it is a prediction that is not difficult to check by experiment. The first confirmation came in 1908, measuring the mass of fast electrons in a vacuum tube. In fact, the electrons in a color TV tube are about half a percent heavier than electrons at rest, and this must be allowed for in calculating the magnetic fields used to guide them to the screen.
>
> Much more dramatically, in modern particle accelerators very powerful electric fields are used to accelerate electrons, protons and other particles. It is found in practice that these particles become heavier and heavier as the speed of light is approached, and hence need greater and greater forces for further acceleration. Consequently, the speed of light is a natural absolute speed limit. Particles are accelerated to speeds where their mass is thousands of times greater than their mass measured at rest, usually called the "rest mass." (Fowler, 1996)

Another textbook states that "…when we consider the motion of a system of particles (such as gas molecules in a moving container), the total mass of the system must be taken to be the sum of the relativistic masses of the particles rather than the sum of their rest masses." (Sears, Zemansky, and Young, 1987, p. 807)

Yet, in the next paragraph Sears et al. (1987) admits the pitfalls of using the relativistic mass. For example, Newton's second law cannot be generalized by its relativistic counterpart as $F = m_{rel}a$, nor can the relativistic kinetic energy of a particle be generalized as KE = ½ $m_{rel}v^2$. It is thus better to consider the momentum expression in Equation (39) as a generalized definition of momentum with m construed as the rest mass, which is a property of the particle on the same par with charge, independent of motion.

An increasing number of textbooks published during 2000-2005 have not relied on the concept of relativistic mass. The reasons are many. In 1948, Einstein in a letter to Lincoln Barnett wrote:

> It is not good to introduce the concept of the mass $M = m/(1-v^2/c^2)^{1/2}$ of a body for which no clear definition can be given. It is better to introduce no other mass than `the rest mass' m. Instead of introducing M, it is better to mention the expression for the momentum and energy of a body in motion.

Einstein himself was ambivalent about the term relativistic mass, which he almost never used, although in a paper of 1905 {"Does the inertia of a body depend on its energy content?") he stated that the inertia of a body depended on its energy. This is in accord with what some scientists say when they popularize the topic for the general public. In Stephen Hawking in *A brief history of time*, we find, "Because of the equivalence of energy and mass, the energy which an object has due to its motion will add to its mass." and Richard Feynman in *The character of physical law* writes, "The energy associated with motion appears as an extra mass, so things get heavier when they move." In the 1920s, physicists Pauli, Eddington, and Born freely used relativistic mass in their textbooks.

Gibbs (1996) probably best summarizes the confusing state of affairs about relativistic mass and rest mass in the following passage:

> There is sometimes confusion surrounding the subject of *mass* in relativity. This is because there are two separate uses of the term. Sometimes people say "mass" when they mean "relativistic mass", m_r but at other times they say "mass" when they mean "invariant mass", m_0. These two meanings are not the same. The invariant mass of a particle is independent of its velocity v, whereas relativistic mass increases with velocity and tends to infinity as the velocity approaches the speed of light c. They can be defined as follows:
>
> $m_r = E/c^2$
> $m_0 = \text{sqrt}(E^2/c^4 - p^2/c^2)$
>
> where E is energy, p is momentum and c is the speed of light in a vacuum. The velocity dependent relation between the two is
>
> $m_r = m_0 /\text{sqrt}(1 - v^2/c^2)$
>
> Of the two, the definition of invariant mass is much preferred over the definition of relativistic mass. These days, when physicists talk about mass in their research, they always mean invariant mass. The symbol m for invariant mass is used without the subscript 0. Although the idea of relativistic mass is not wrong, it often leads to confusion, and is less useful in advanced applications such as quantum field theory and general relativity. Using the word "mass" unqualified to mean relativistic mass *is* wrong because the word on its own will usually be taken to mean invariant mass. For example, when physicists quote a value for "the mass of the electron" they mean its invariant mass.

By 1950s with the rise of particle physics more and more scientists began to shun the term relativistic mass, and whenever they referred to mass they meant invariant mass. Certainly it makes no sense to refer to the rest mass of a photon as zero since photons are never at rest. Photons are simply massless. Although photons have momentum, which is related to total energy, the concept of relativistic mass has no significance to photons.

2.7 Relativistic Energy

If there is one equation that dominates the twentieth century among physicists and in the minds of the public, it is the mass-energy equivalence relation $E = mc^2$. Simple as it is, Einstein's formula has far-reaching consequences. It ushers in the atomic age by showing that a small amount of mass is equivalent to an enormous amount of energy. The trick is to find the technology to effect the conversion. One peaceful application of this insight is evident in nuclear power plants, where small amounts of uranium produce by fission tremendous heat energy that is used to generate electricity using conventional turbines. The sun too emits vast quantities of energy by burning tiny amounts of its own mass. Conversely, energy has been converted into electrons and protons in the laboratory.

Two of the most fundamental concepts in physics are conservation of energy and conservation of momentum. We see these laws of conservation at work in collisions and other chemical and physical phenomena. We shall see the same laws applied in deriving the relationship between mass and energy.

In what follows Jones and Childers (1993, p. 708) introduce Einstein's famous equation by way of momentum. Maxwell's electromagnetic theory predicts that light exerts pressure when it strikes an object. This pressure implies that light rays carry momentum, which is proportional to the energy carried by the light rays. The momentum is expressed by

$$p = E / c \qquad (41)$$

where E is the energy and c the speed of light.

Imagine, in a thought experiment by Einstein, a closed rectangular box of length L and mass M suspended by a wire to allow free, frictionless motion. At an instant of time a series of flashbulbs on one end of the box fires a flash of light, causing the box to recoil by a certain distance x with an equal but opposite momentum to that of the lights, so that we have:

$$p_{box} = Mv = p_{light} = E / c \qquad (42)$$

where v is the velocity of the box's recoil and E the energy of the light.

The light travels during time $t = L / c$ to reach the opposite end of the box, where it is absorbed while the box moves a distance $-x$ in its recoil. At the end of the light's travel the system returns to its rest state, thus conserving momentum.

During the movement of the box, its center of mass must stay stationary. For this to be the case there must be an comparable shift caused by an equivalent mass m associated with the light such that the movement of m over the distance L is compensated by the movement of the box's mass M over the distance $-x$, as indicated by

$$mL = Mx \qquad (43)$$

Since the momentum of the box and that of the light are equal, we can solve for the velocity of the box by using relation (42):

$$v = E / Mc \qquad (44)$$

Given that the elapsed time t is the time during which the light travels the distance L, we can derive the distance traveled by the box as follows:

$$x = vt = (E / M\,c)\,(L / c) \qquad (45)$$

Substituting the value of x in equation (43) in equation (45), we get:

$$mL / M = (E / M\,c)\,(L / c)$$

Eliminating L and M from both sides, we derive the **mass-energy equivalence relation** as:

$$\boldsymbol{E = mc^2} \qquad \textbf{(46)}$$

SI unit: J.

In the above Jones and Childers passage the principle of conservation of momentum plays a crucial role.

The mass-energy relation equation says that energy is equal to mass times the speed of light squared. First it means that energy and mass are equivalent, i.e., they are just two forms of the same thing. Second, since the speed of light is a huge number, a small amount of mass can produce an enormous amount of energy. We have seen this in nuclear reactions.

Example 42: If a 1-kg grapefruit could be converted directly into electromagnetic energy, how much energy is obtained in J and in MeV?

Solution: We note that no object can be completely converted into energy. Applying the relation (46) we obtain the energy that could theoretically be released even though current technology falls far short of the feat:

$$E = mc^2 \qquad (46)$$

$$= (1.00 \text{ kg})\,(3.00 \times 10^8 \text{ m/s})^2 = \underline{9.00 \times 10^{16} \text{ J.}}$$

or, since 1 MeV = 1.602×10^{-13} J,

$E = 9.00 \times 10^{16}$ J / 1.602×10^{-13} J = $\underline{5.62^{29}\ MeV.}$

This is enough energy to run a 1-kW motor for 2.85 million years. Given that 1 kilowatt-hour is equal to 1000 J/s x 60s/min x 60 min/h = 3.60 MJ, we get: $E = 9.00 \times 10^{16}$ J / $3.60 \times 10^{8} = 2.5 \times 10^{10}$ kW/h. Since there are roughly 24 h/day x 365 day/y = 8760 h/y, this energy, emitted at 1 kW/h, will last 2.5×10^{10} kW/h / 8760 h = 2.85×10^{6} years.

Kinetic Energy

Kinetic energy is energy produced by an object in motion. Classically, it is $KE = 1/2\ mv^2$, that is the kinetic energy of an object of mass m moving with speed v. This formula works only where $v << c$. Hence, just as Galilean relativity is a special case of special relativity, so is classical kinetic energy a special case of relativistic kinetic energy. We will now redefine the concept of kinetic energy in order to accommodate objects moving at slow as well as fast speeds.

Recall the relativistic mass relation:

$$m = m_0 \ / \ (1 - v^2 / c^2)^{1/2} \qquad (40)$$

By applying the binomial theorem, expanding γ, we get:

$$m = m_0\ (1 + ½\, v^2/\, c^2 + 3/8\ (v^2/\, c^2)^2 + \ldots - 1)$$

Multiplying both sides by c^2 and keeping only the first two terms of the series because the remaining terms are too small to make a difference , yields:

$$mc^2 = m_0\, c^2 \ + ½\, m_0\, v^2$$

Noting that $E = mc^2$, the left-hand term is total energy, the first term of the right-hand side is energy related to the rest mass, and the second term is its classical KE, we have the **total energy** equation:

$$\boldsymbol{E = m_0\, c^2 + KE} \qquad \textbf{(47)}$$

Rearranging the terms, we can say that **the kinetic energy of an object is the difference between its total energy *E* and its rest energy $m_0\, c^2$**:

$$\boldsymbol{KE = mc^2 - m_0\, c^2} \quad \text{or} \quad \boldsymbol{KE = E - m_0\, c^2} \qquad \textbf{(48)}$$

Since $m = m_0\ /\ (1 - v^2 / c^2)^{1/2}$ (equation 40), we can also write the kinetic relation as:

$$\boldsymbol{KE = m_0 c^2\ /\ (1 - v^2 / c^2)^{1/2} - m_0\, c^2} \qquad \textbf{(49)}$$

If the object is at rest, its kinetic energy is zero and its total energy becomes its **rest energy**:

$$\boldsymbol{E_0 = m_0\, c^2} \qquad \textbf{(50)}$$

And if we use the relativistic mass equation (40), the total energy can be expressed by:

$$\boldsymbol{E = m_0\, c^2\ /\ (1 - v^2 / c^2)^{1/2}} \quad \text{or} \quad \boldsymbol{E = \gamma\, m_0\, c^2} \qquad \textbf{(51)}$$

This and previous formulas imply equivalence of mass and energy. The interconversion between mass and energy has been amply demonstrated in the laboratory and in nuclear power plants. Electricity is generated in nuclear plants with energy released when the nuclei of uranium fuel split into two nuclei and one or two neutrons, which in turn create more fission and more neutrons in a chain reaction. The process results in a loss of small amounts of uranium mass and a release of immense kinetic energy which turns water into steam which drives turbines to generate electricity. In a reverse process electromagnetic radiation can be converted into matter such as electrons in the laboratory. Also in pair-production a photon is converted into an electron and a positron with kinetic energies released.

Example 43: Find the total energy and kinetic energy in J and in MeV of an electron that moves with a speed of 0.70 c and whose mass is 9.11 x 10^{-31} kg.

Solution: Applying equation (51) we calculate the total energy:

$$E = m_0\, c^2 \;/\; (1 - \, v^2 / c^2)^{1/2} \qquad (51)$$

$$E = (9.11 \text{ x } 10^{-31} \text{ kg})\, (3.0 \text{ x } 10^8 \text{ m/s})^2 \;/\; (1 - \, 0.70^2\, c^2 / c^2)^{1/2}$$

$$= \; 8.2 \text{ x } 10^{-14} \text{ J x } \; 1.4 \; = \; \underline{11.48 \text{ x } 10^{-14} \text{ J.}}$$

$$= \; 11.48 \text{ x } 10^{-14} \text{ J} \,/\, 1.60 \text{ x } 10^{-10} \text{ J} = \; \underline{0.7175 \;\; \text{MeV.}}$$

The kinetic energy is equal to the difference between the total energy and the rest energy:

$$KE \; = \; E - m_0\, c^2 \qquad (48)$$

$$= \; 11.48 \text{ x } 10^{-14} \text{ J} \; - \; 8.2 \text{ x } 10^{-14} \text{ J} = \underline{3.28 \text{ x } 10^{-14} \text{ J}}.$$

$$= \; 3.28 \text{ x } 10^{-14} \text{ J} \,/\, 1.60 \text{ x } 10^{-10} \text{ J} = \; 0.\underline{205 \;\; \text{MeV.}}$$

Energy is often expressed in terms of MeV and mass is expressed in MeV/c^2 (without carrying out the division) or in kg. Since mass and energy are just two faces of the same coin, the interconversion between them makes it possible to express them in either unit of measurement. For example,

$$1 \text{ kg} \;\leftrightarrow 5.609\,5 \;\text{ x }\; 10^{29} \;\text{ MeV}$$

The following table lists the rest mass and rest energy of a number of common particles.

Masses and Rest Energies of Common Particles and Atoms

Particle	**Mass *m* (kg)**	**Rest Energy mc^2 (MeV)**
Photon	0	0
Neutrino	0	0
Electron or positron e	9.109 389 7 x 10^{-31}	0.510 999
Muon $\mu^{\pm}$	1.883 333 x 10^{-28}	105.658
Pi meson (neutral) π^0	2.405 95 x 10^{-28}	134.964
Pi meson (charge) $\pi^{\pm}$	2.488 05 x 10^{-28}	139.569
Pion (+)	2.416 5 x 10^{-28}	135.56
Atomic mass unit u	1.660 540 x 10^{-27}	931.494
Proton p	1.672 623 x 10^{-27}	938.272
Neutron n	1.674 929 x 10^{-27}	939.565
Deuteron d or ^2H	3.343 584 x 10^{-27}	1857.612
Triton	5.007 357 x 10^{-27}	2808.920
Alpha particle α or ^4He	6.644 72 x 10^{-27}	3727.41
Hydrogen atom	1.673 534 x 10^{-27}	938.783
Deuterium atom	3.344 497 x 10^{-27}	1876.12

Table 2-4. The table of masses and energies shows the correspondence between them: A larger mass corresponds to a larger rest energy. These values facilitate the calculations of relativistic energy.

In situations where the momentum and energy are known but not the velocity, it is convenient to use a total energy relation that does not depend on speed. Given $E = mc^2$ and $p = mv$ where $m = m_0 \,/\, (1 - v^2 / c^2)^{1/2}$, we square both sides of the total energy equation and write the **total energy relation in terms of momentum**:

$$E^2 = m^2c^4 = m^2c^2(c^2 + v^2 - v^2)$$

$$E^2 = m^2c^2v^2 + m^2c^2(c^2 - v^2)$$

Substituting $m_0 \,/\, (1 - v^2 / c^2)^{1/2}$ for m in the second term, we obtain:

$$E^2 = p^2c^2 + m_0^2c^4(1 - v^2/c^2) \,/\, (1 - v^2/c^2)$$

$$\mathbf{E^2 = m_0^2c^4 + p^2c^2} \qquad \textbf{(52)}$$

or

$$\mathbf{E^2 = E_0^2 + (pc)^2} \qquad \textbf{(53)}$$

The formula (52) says that when the object is at rest, its momentum is zero and the equation becomes the familiar $E = m_0c^2$. And when its total energy is much greater than its rest energy, $E >> mc^2$, the first term is negligible and can be omitted to yield the **high-energy equation**:

$$\mathbf{E = pc} \qquad \textbf{(54)}$$

The relation (54) also shows that since the total energy depends solely on momentum, the particle must have zero mass. Such is the case with photons, neutrinos and gravitons. Of the last two, neutrinos have been identified experimentally whereas gravitons are hypothesized but not confirmed. Furthermore, equation (52) shows that total energy varies in direct proportion to momentum. Because of c a small change in momentum results in an enormous change in total energy.

From relation (53) we can derive a useful formula to find **the momentum in terms of total energy and mass**:

$$E^2 = E_0^2 + (pc)^2 \qquad (53)$$

$$\mathbf{p = (E^2 - E_0^2)^{1/2} \,/\, c} \qquad \textbf{(55)}$$

Another useful relation concerns velocity, which can be derived from energy alone. Since $E = \gamma E_0 = E_0 \,/\, (1 - v^2 / c^2)^{1/2}$, we derive **velocity in terms of *c*** thus:

$$E = E_0 \,/\, (1 - v^2 / c^2)^{1/2}$$

Squaring both sides, we get

$$E^2 = E_0^2 \,/\, (1 - v^2 / c^2)$$

$$1 - v^2 / c^2 = E_0^2 \,/\, E^2$$

$$1 - E_0^2 / E^2 = v^2 / c^2$$

$$v^2 = c^2 (1 - E_0^2 / E^2)$$

$$\mathbf{v = (1 - E_0^2 / E^2)^{1/2}\; c} \qquad \textbf{(56)}$$

Example 44: With a mass of 938.27 MeV/c^2, a proton has a kinetic energy of 210 MeV. Find the proton's total energy, momentum, and speed.

Solution: Knowing rest mass and kinetic energy we use equation (47) to find the total energy:

$$E = m_0 c^2 + KE \qquad (47)$$

$$E = (938.27 \text{ MeV/c}^2)\, c^2 + 210 \text{ MeV} = \underline{\mathit{1148.27\ MeV}}.$$

The momentum may then be calculated from equation (55)

$$p = (E^2 - E_0^2)^{1/2} / c \qquad (55)$$

$$p = (\underline{(1148.27 \text{ MeV})}^2 - (938.27 \text{ MeV})^2)^{1/2} / c = \underline{\mathit{661.9\ MeV / c}}.$$

To find the speed, we use equation (56):

$$v = (1 - E_0^2 / E^2)^{1/2}\ c \qquad (56)$$

$$v = (1 - (938.27 \text{ MeV})^2 / (1148.27 \text{ MeV})^2)^{1/2}\ c = \underline{\mathit{0.576\,4\ c.}}$$

<u>Example 45</u>: A proton has a kinetic energy equal to its rest energy. Find the proton's momentum and speed.

<u>Solution</u>: Knowing rest energy (See Table 2-4 above) and kinetic energy are equal we use equation (47) to find the total energy:

$$E = m_0 c^2 + KE \qquad (47)$$

$$E = 938.27 \text{ MeV} + 938.27 \text{ MeV} = 2 \text{ x } 938.27 \text{ MeV}.$$

The momentum is:

$$p = (E^2 - E_0^2)^{1/2} / c \qquad (55)$$

$$p = 938.27 \text{ MeV } (4 - 1)^{1/2} / c = \underline{\mathit{1625\ MeV / c.}}$$

And the speed is:

$$v = (1 - E_0^2 / E^2)^{1/2}\ c \qquad (56)$$

$$v = (1 - 938.27^2 \text{ MeV} / (2 \text{ x } 938.27)^2 \text{ MeV})^{1/2}\, c$$

$$v = (1 - 1/4)^{1/2}\, c = \underline{\mathit{0.866\ c}}.$$

<u>Example 46</u>: If an electron moves with speed 0.8 c, find its total energy, its kinetic energy, and its momentum.

<u>Solution</u>: Using Table 2-4, we get $m_0 c^2 = 0.510\,999$ MeV. To find the total energy, we use equation (51):

$$E = m_0 c^2 / (1 - v^2 / c^2)^{1/2} \qquad (51)$$

$$E = 0.511 \text{ MeV} / (1 - 0.8^2 / c^2)^{1/2} = \underline{\mathit{0.852\ MeV.}}$$

Its kinetic energy is given by equation (48):

$$KE = E - m_0 c^2 \qquad (48)$$

$$KE = 0.852 \text{ MeV} - 0.511 \text{ MeV} = \underline{\mathit{0.341\ MeV}}.$$

The momentum is then given by equation (55):

$$p = (E^2 - E_0^2)^{1/2} / c \qquad (55)$$

$$p = (0.852^2 \text{ MeV} - 0.511^2 \text{ MeV})^{1/2} / c = \underline{0.681 \text{ MeV} / c.}$$

We can also obtain the momentum by using the relativistic momentum formula (38).

$$p = \gamma m v \qquad (38)$$

$$p = mv / (1 - v^2 / c^2)^{1/2} \qquad (38)$$

$$= (0.511 \text{ MeV}) (0.8\, c) / (1 - 0.8^2\, c^2 / c^2) = \underline{0.681 \text{ MeV} / c.}$$

2.8 Relativistic Doppler Effect

Described by the Austrian physicist Christian Doppler (1805-1853) in 1842, the Doppler effect on sound waves occurs when we hear a high pitch of a fire truck's siren approaching us and a lower pitch as the truck recedes from us. This is because the sound waves are compressed to a higher frequency (or shorter wavelength) in front of the truck than they are behind it. The same effect is felt when the wave source is stationary and the observer is moving toward or away from it, or both source and observer are moving toward or away from each other. In 1849 the French physicist Armand Hippolyte Louis Fizeau (1819-1896) applied the shift to light waves. The **Doppler effect** (also known as **Doppler-Fizeau effect**) applies to all waves, sound, electromagnetic waves, even water. It is used in radar to measure the speed of cars and airplanes.

Significant differences exist between the Doppler effect for sound and that for light. First, if an object approaches an observer at greater than the speed of sound, she will not hear anything until the sonic boom hits. In contrast, nothing travels faster than light. Then, the Doppler effect for light depends entirely on the velocity of the emitter relative to that of the receiver. The effect for sound depends on the velocities of both emitter and receiver relative to air. Finally, the Doppler effect occurs when the object is in the line of sight only. If the light signal travels along a line perpendicular to the line of sight, we have a **transverse Doppler shift**. Suppose a train is traveling along a straight railroad track far away from you. You would not hear any Doppler shift for the sound. But for light, you would see a frequency decrease equal only to the time dilation factor, but no effect due to the motion of the train.

In the case of light emission, when the observer moves toward the light source (for example, a star or a galaxy) or vice versa along the line of sight, the light's frequency intensifies and shifts toward the blue (or shorter wavelength) end of the spectrum. This is called a **blueshift.** When the star or galaxy moves away from the observer, the light emitted shifts toward the lower frequency (longer wavelength) or red end of the spectrum and we have a **redshift**. As stated above, there is no Doppler effect if the light is perpendicular to the line of sight. In this case there is only the time dilation effect but no Doppler effect. Astronomers use the Doppler effect as an important tool to determine the **radial velocities** (speeds of the celestial bodies in their motion parallel to our line of sight) and motions of planets, stars and galaxies. Since most galaxies and stars are known to be redshifted, the conclusion is that they are moving away from the Earth and hence, that the universe is expanding.

The equation that expresses **the nonrelativistic Doppler shift** is simple:

$$\boldsymbol{\Delta\lambda / \lambda_0 = v / c} \qquad \textbf{(57)}$$

where $\Delta\lambda = \Delta\lambda - \Delta\lambda_0$ is the difference in wavelength between the known rest wavelength $\Delta\lambda_0$ and that observed $\Delta\lambda$, and v the radial velocity. This equation works only if $v << c$.

Example 47: In a study of the spectrum of the star Vega, a hydrogen line Hα is observed to have a wavelength of 656.255 nm. This line has a normal wavelength of 656.285 nm. Find the star's wavelength shift and radial velocity.

Solution: The star's wavelength shift is:

$$\Delta\lambda - \Delta\lambda_0 = 656.255 \text{ nm} - 656.285 \text{ nm} = -\,\underline{0.030 \text{ nm.}}$$

We derive its radial velocity from equation (57):

$$\Delta\lambda / \lambda_0 = v / c$$

$$v = (\Delta\lambda / \lambda_0)\, c = (-0.030 \text{ nm} / 656.285 \text{ nm}) \times 3.0 \times 10^8 \text{ m/s} = -\underline{13.7 \text{ m/s.}}$$

The minus sign indicates that Vega is moving toward us.

At very high velocity, we encounter a more complicated **relativistic Doppler shift** because of time dilation. Imagine a laser beam mounted on a spaceship moving at speed v toward an Earth observer. The laser has frequency f_0 in the reference frame in which the ship is at rest. During each time period T of the light beam the spaceship has traveled the distance vT and the light has covered cT. Due to time dilation, T appears to run slower to the observer on Earth than to the astronauts in the spaceship, who measure the proper period T_0. The period T_0 between emissions is $T_0 = 1 / f_0$. As the spaceship approaches the observer, its motion compresses the wavelength of the light increasing the frequency being observed. Recalling that the period $T = \gamma T_0$, the wavelength is thus:

$$\lambda = cT - vT = (c - v)\, T_0 / (1 - v^2 / c^2)^{1/2}$$

The wave equation gives the frequency f measured by the Earth observer:

$$f = c / \lambda$$

or , using $f_0 = 1 / T_0$:

$$f = f_0 (1 - v^2 / c^2)^{1/2} / 1 - v / c$$

Expanding the square root of the numerator and rewriting the denominator, we have:

$$f = f_0 [(1 + v / c) (1 - v / c)]^{1/2} / [(1 - v / c) (1 - v / c)]^{1/2}$$

Simplifying, we get the frequency of the light approaching the observer:

$$\mathbf{f = f_0 [(1 + v / c) / (1 - v / c)]^{1/2}} \qquad \textbf{(approaching)} \qquad \textbf{(58)}$$

In this case, as the source is moving toward the observer, its frequency increases (or equivalently, its wavelength decreases). If the source and the observer move apart, the speed v becomes negative and the frequency decreases as shown below.

$$\mathbf{f = f_0 [(1 - v / c) / (1 + v / c)]^{1/2}} \qquad \textbf{(receding)} \qquad \textbf{(59)}$$

Again what matters in the relativistic Doppler effect for light is the relative speed of the source and observer.

Example 48: A rocket ship is traveling toward a distant star that emits yellow light at the frequency of 5.0×10^{14} Hz measured in the star's rest frame. If the ship's velocity relative to the star is 0.15 c, what frequency does the rocket ship's crew observe for the star's light?

Solution: We apply formula (58):

$$f = f_0 [(1 + v / c) / (1 - v / c)]^{1/2} \qquad (58)$$

$$= 5.0 \times 10^{14} \text{ Hz} [(1 + 0.15\, c / c) / (1 - 0.15\, c / c)]^{1/2} = \underline{5.82 \times 10^{14} \text{ Hz.}}$$

Converted to wavelength, the above frequency is:

$$\lambda = c / f$$

$$\lambda = 3.0 \times 10^8 \text{ m/s} / 5.82 \times 10^{14} \text{ Hz} = \underline{516 \text{ nm}}.$$

This corresponds to green light.

By measuring the Doppler shift of the light of stars and galaxies astronomers have determined that its frequency decreases as compared with line spectra maintained in the laboratory. That is, it has redshifted to the longer wavelength end of the spectrum. Thus stars and galaxies are moving away from Earth, and the universe is expanding. Furthermore, the observed red shift seems proportional to their distance from us, i.e., the farther away the celestial objects are, the faster they recede from Earth, in accord with Hubble's Law.

For red shifts larger than a few tenths, the Doppler formula (57) gives the incorrect answer. For example, a redshift of 3.0 applied to a quasar would indicate that the quasar is moving away at three times of speed of light in violation of the second postulate of special relativity. Therefore, the redshift expression has to be modified as follows:

$$\boldsymbol{z = \Delta\lambda / \lambda_0 = [(1 + v / c) / (1 - v / c)]^{1/2} - 1} \qquad \textbf{(60)}$$

where z is the symbol for the redshift.

As an example, astronomers observed a redshift of 3.2 in the series of hydrogen emissions from a quasar that were Doppler shifted into the red region of the spectrum. Applying the formula (60), we get:

$$z = \Delta\lambda / \lambda_0 = [(1 + v / c) / (1 - v / c)]^{1/2} - 1 \qquad (60)$$

$$3.2 = [(1 + v / c) / (1 - v / c)]^{1/2} - 1$$

$$(1 + v / c) / (1 - v / c) = (3.2 + 1)^2 = 17.6$$

$$(1 + v / c) = (17.6) (1 - v / c)$$

$$18.6\, v / c = 16.6$$

$$v / c = 0.89.$$

Thus a redshift of 3.2 means that the quasar is moving away from Earth at the speed of 0.89 c.

3. General Relativity

We have seen that special relativity applies to laws of physics in inertial or non-accelerating reference frames. In 1915 Einstein introduced the general theory of relativity to accommodate all reference frames, including accelerating ones. Basically the general theory of relativity is a theory of gravity. Whereas Newtonian physics focuses on force, general relativity focuses on motion. Though both approaches yield satisfactory results at low everyday speeds, relativity alone is adequate to account for the behavior of objects moving at high speeds and having large gravitational fields. We will first discuss the classical dichotomy inertial mass-gravitational mass to see how it leads to the principle of equivalence. We will see the principle of covariance, which together with the equivalence principle lays the foundation of the general theory of relativity. Finally, we will discuss the various experimental evidences that support the theory and the implications of general relativity in astronomy and cosmology.

3.1 Inertial Mass and Gravitational Mass

In Newtonian mechanics, a distinction exists between **inertial mass** that resists acceleration and **gravitational mass** that exerts force between objects. Although Newton believed inertial mass and gravitational mass are the same, he measured them differently. Newton's first law that states that a body at rest or in motion at a constant velocity in a straight line tends to keep its rest position or its motion unless acted on by an outside force. This law and the **second law** below apply to inertial mass.

$$F = m_i a \qquad \textbf{(61)}$$

where F is the net force exerted on an inertial mass m_i to affect its inertia or motion, and a is acceleration.

But **weight** is a force of gravity that acts on a body as shown in:

$$W = m_g g \qquad \textbf{(62)}$$

where W is the force exerted by gravity on gravitational mass m_g, and g is the acceleration of gravity on Earth with an average value of $g \approx 9.80$ m/s^2.

Inertial mass m_i is defined as a measure of the inertial property of matter. In practice, it is easier to determine mass, not by observing its interaction with another object of known mass, but by measuring the force of gravity on the object. We know that when the *only* force around is gravity all objects regardless of mass (or weight} fall with the same acceleration g. If we now call g the acceleration of free-falling objects and W the force of gravity (which is also called weight), then we can derive the second equation (62) from Newton's second law (equation 61).

From equations (61) and (62), we can write:

$$a = m_g / m_i \, g \qquad \textbf{(63)}$$

If acceleration is constant for a gravitational field, which it is, then the ratio of gravitational to inertial mass is also constant. And if acceleration is chosen to be equal to the force of gravitation, as it is above, then we can say that gravitational mass and initial mass are equal. Given this we can also say that inertia and weight are just two manifestations of the same thing.

We note in passing that the second equation makes clear the fact that weight is not the same as mass. Mass in one definition is the 'quantity of matter' in a body or more precisely the measure of the inertia of a body. Mass is also regarded as an inherent property of matter, so that an object may be weightless (where g = 0) but still has mass. On the moon, where gravity is one-sixth that of the earth, objects weigh one-sixth their weight on earth even though their mass remains the same. There are a few exceptions: Photons, neutrinos and gravitons are massless particles.

Newton's insight is to see the same force acting in the fall of an apple to the ground and keeping the moon orbiting around the earth. This unified view is embodied in the formulation of the **law of universal gravitation** that governs the attraction between any two bodies in the universe, from particles to galaxies:

$$F = G\, m_1 m_2 / r^2 \qquad \textbf{(64)}$$

where F is the force of attraction, G is the constant of proportionality called gravitational constant, which is equal to 6.672 x 10^{-11}N.m^2/kg^2, m_1 is the gravitational mass of the first object, m_2 that of the second object, and r the distance between the centers of gravity of the objects.

Both the apple and the moon are attracted to the center of the earth by the same gravitational force, in this case a centripetal one. If there were no centripetal force to keep the moon in its nearly circular orbit, it would follow its straight inertial path into space. From the law of gravitation, we can say first that all bodies in the universe attract (not repel) all other bodies. Second, the magnitude of the gravitational force is directly proportional to the gravitational masses of the bodies involved, e.g., if the mass of one body doubles, the gravitational force doubles. Third, the gravitational force is inversely proportional to the square of the distance that separates them, e.g., the gravitational force is reduced by half if the distance squared (not the simple distance) is doubled. Finally, the **gravitational constant *G*** is a universal constant that had to be determined experimentally. Since G is so small the gravitational force between any two objects encountered in everyday life is tiny. As an example, for two 3000-kg trucks whose centers of gravity are 3 m apart, their force of mutual attraction is only about 67 μN (micronewton or millionth of a newton, a newton being a unit of force N = 1kg.m/s^2).

Gravitational Constant G

Of the four forces of nature, the strong, weak, electromagnetic and gravitational forces, the last one is the weakest. It is also very difficult to measure. Among the difficulties may be cited the errors inherent in the measuring apparatus, such as inconstancy of the torsional moment of suspension in torsion balance and torsion pendulum devices and the inhomogeneities of the masses used in the apparatus. Others include environmental factors such as the disturbances of the ground, the effects of ambient temperature, the magnetic and electric influences, the gradients of the gravitational field, and the changes of the surrounding field.

Over the last two hundred years a number of scientists have measured G, among whom the French physicist Charles de Coulomb (1736-1806), the English physicist Henry Cavendish (1731-1810), who used the torsion balance, and others up to modern times. Some physics classes even have students conduct measurements of G as a project. Most textbooks settle on the value of G = 6.67259 x 10^{-11}N.m^2/kg^2 ± 0.00085 recommended by the CODATA (Committee on Data for Science and Technology of the International Council of Scientific Unions). However, attempts have been made in recent years to refine the measurement of G, which some scientists think is still incorrect. More modern apparatuses have been used with varying results so that it is reasonable to expect the controversy to continue.

3.2 The Principles of Equivalence and Covariance

We have seen that the two gravitational and inertial masses have been shown experimentally to be virtually equal, to within a few parts in 10^{11}: One is the force of mutual gravitational attraction among masses (m_g) and the other the resistance of a single mass to acceleration (m_i). With his general theory Einstein showed the fundamental connection between them, i.e., there is equivalence between them and their effects. We will now see how Einstein uses this correspondence to set forth the general theory of relativity.

In a thought experiment proposed by Einstein (p. 66) an observer is inside a closed chest the size of a room suspended in space, which during the next step will be moved upward by an outside force unbeknownst to the observer. We may modernize the experiment by substituting an elevator or a rocket ship for the chest, if we wish, but the result will be the same.

Before proceeding further let us understand the concept of gravitational field. It is common knowledge that two things separated in space cannot interact without some sort of intermediate agent. There is no such thing as action at a distance. An apple in the tree does not fall to the ground without what we now call gravity. Just as a magnet attracts iron filings because it is surrounded by a magnetic field, so does the earth attract the apple and the moon because it is surrounded by a gravitational field. The closer to earth an object is, the stronger it is affected by the earth's gravitational force. Far out in space, the earth's gravitational field diminishes in magnitude. Under the influence of the gravitational field all objects are subjected to the same acceleration regardless of their mass, material, or physical state. A lump of lead and a feather fall with exactly the same speed in the same gravitational field in vacuum.

Now in the closed chest mentioned above, there is no gravitational field acting on the system. Suppose that without the observer's knowledge the chest is lifted by an outside force by means of a rope through a hook fastened to the top of the chest, with an acceleration equal to g = 9.8 m/s^2. The acceleration of the chest moving upward transmits its force to her legs, which press down on the chest's floor with an equal force, so that the net result is that she stands on the chest floor exactly as she would in her living room without feeling any difference. If the observer drops a rubber ball from her hand, the acceleration of the chest is no longer transmitted to the ball, and it approaches the floor with accelerated motion. She also notices that the acceleration of the ball toward the floor is the same if she used a heavy weight. The circumstances just described illustrate the effect of acceleration created by an external force (rocket firing, angels lifting, or whatever) on inertial mass. Suppose now that the chest is placed back on earth, again without the observer's knowledge. The earth's gravitational force is transmitted to her legs, and she stands on the floor in exactly the same way she would in her own living room. If she drops the ball, it falls to the ground with the same accelerated motion as observed above. Here is an example of the effect of a gravitational field on gravitational mass. In both cases, the observer observes the same behavior of the ball. Hence, inertial mass and gravitational mass are equal in the sense that they behave in identical ways. Furthermore, acceleration and gravitational field produce identical effects.

In fact, there are no experiments, mechanical or otherwise, that allow the observer to distinguish acceleration due to gravitational field from acceleration due to forces of other kinds. This leads us to the **principle of equivalence**, which is one of the postulates of the general theory of relativity.

The principle of equivalence:

Experiments performed in an inertial frame containing a gravitational field and experiments performed in a uniformly accelerating frame yield identical results.

The other postulate is known as the **principle of covariance**, which states:

All physical laws can be formulated to be valid for any observer in any spacetime reference frame whether accelerated or not.

This principle calls for the formulation of relativity equations in any system of spacetime coordinates. There is still ongoing controversy over the covariance principle, and over the consistency of covariance and equivalence, but this is beyond the scope of this work. Suffice it say that the two principles of equivalence and of covariance form the foundation of general relativity.

As a theory of gravitation the general theory of relativity does away with the notion of mass-dependent force, which Newtonian physics relies on to explain gravitation. Instead, relativity focuses on motion and makes use of the concept of spacetime and the geometry of space. With this new perspective gravitation is nothing but a consequence of the spacetime curvature. We will next look at the test of the general theory of relativity.

The Curvature of Spacetime

Before going further, let us briefly familiarize ourselves with spacetime curvature, which is one of the startling insights of general relativity. According to this theory, we live in a world with four dimensions, three space dimensions and one time dimension, in one continuum, unlike the Newtonian world, where space and time are two separate entities. We will treat this subject in more detail later, but for now a preliminary introduction is needed to aid in comprehending the evidence presented below.

Einstein maintains that gravity is just a consequence of the curvature of spacetime. In other words, it is the geometry of space that dictates the behavior of objects. Space is not flat but curved around celestial bodies. Almost all bodies warp the space around them by virtue of their mass. The more mass a body has, the more distortion of surrounding space it produces. Imagine a piece of fabric that is held up and kept taut by hooks attached to all its corners. If we place a bowling ball on the fabric, it will roll to the center and make a depression where it rests. The surface of the fabric around the bowling ball is curved inward like a funnel. If we roll a marble on the fabric toward the bowling ball, it will first follow a straight path when still far from the ball, then starts to dip when it gets closer to the ball. The marble merely follows the curvature of the fabric around the bowling ball. Like the bowling ball the sun warps the space around it and any light that comes near it will behave in the same way as the marble does in the vicinity of the ball. The light naturally follows a geodesic, the shortest distance between two points, through space. Far away from any mass, light propagates in a straight line just like in Euclidean (flat) space. But when it reaches the curved space around a massive body such as the sun, a star or a galaxy, it takes the curved path, i.e., it bends.

3.3 Tests of the General Theory of Relativity

The first test is the **bending of light**. General relativity predicts that light bends in the vicinity of a body with a large mass because of the curvature of spacetime it produces. Let us take a thought experiment in which an observer is inside an elevator at rest in the gravitational field of the earth. She shines a light horizontally and it strikes the opposite wall at the same level to her naked eye. The bending effect is imperceptible because it is too small. Yet in an analogous situation when she tosses a ball it follows a curved path. Suppose now that without her knowledge the elevator starts moving with acceleration equal to *g*. As before if she sends a light beam across the elevator, she will see a straight beam with her own unaided eyes. By the principle of equivalence between gravitational field and acceleration, we should get the same experimental result. Hence, if the elevator's width is 3 m, the light traverses it in 10^{-8} s. During this time the elevator would have moved up 5 x 10^{-16} m to meet the light, and the light beam should hit the wall below the horizontal by the same distance. In either case, the deflection of light is imperceptible. We need a larger scale and a stronger mass to augment the bending and make it measurable.

To that effect we use the sun for its great mass and the space warp it creates around it. Consider a ray of light from a distant star. As it approaches the sun, the curvature of space near its surface deflects the light ray from its straight-line path toward Earth (Figure 3-1). It is as if the sun acts as a **gravitational lens** that refracts light. An earth observer sees the star as if it is in the line of sight. Actually, she merely sees the refraction of the star. The light deflection is predicted to be small, however, only 1.75 arc-seconds (1 arc-second = 1/3600 of a degree) because the Sun's gravity is weak by relativistic standards.

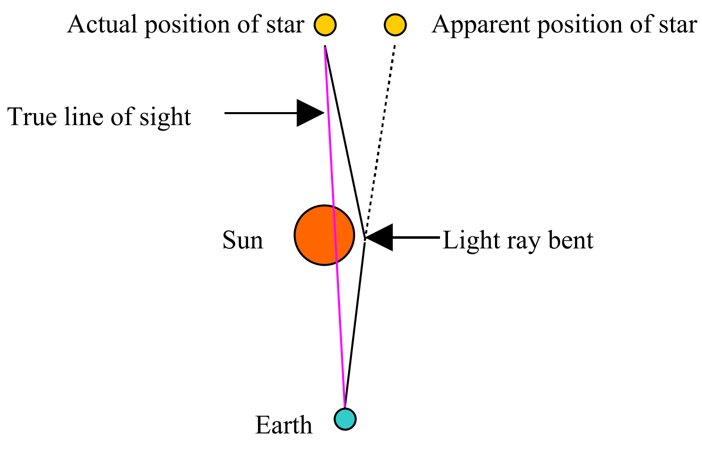


Figure 3-1. Light Deflection Due to Gravitational Field of the Sun
The light rays from the star are deflected from their straight-line paths by the Sun. Since the Sun's gravity is weak, the deflection is only 1.75 arc-seconds. The figure is exaggerated to show light bending.

To confirm the prediction of general relativity, scientists first took photographs of the stars near the sun when it was in total eclipse by the moon to avoid its blinding light. Six months later, when the sun was on the other side of the earth, scientists took photographs of the same stars again. They then compared the positions of the stars in the photographs. In an expedition to Africa in 1919 during a total solar eclipse the English astronomer Sir Arthur Eddington (1882-1944) confirmed the above results. The deflections measured less than 2 arc sec in agreement with theoretical predictions. This success leads to an entirely new view of the structure of spacetime. In contrast, Newtonian mechanics cannot predict light deflection because light has no mass and its calculations of gravitational force depend on mass. In other tests with radio waves, astronomers using interferometry accurately measure the deflection of radio waves to within 10 percent of the value predicted by the theory. In cases where a huge mass such as a galaxy act as a gravitational lens between the earth and the distant star, it bends the light's path so as to make it visible to an earth observer. If the mass has circular symmetry, the gravitational lens effect creates a ring of light. An example of this is Einstein's Ring, a double-lobed radio galaxy formed by a lensing galaxy along the line of sight. If the mass has irregular shape, the light is broken into patches, such as in quasar G2237+0305, called Einstein's Cross, which has four patches around a central glow. In general, since the curvature called for by general relativity is so slight in the solar system, Newtonian mechanics yields just as good results as Einstein's theory.

Another test of which general relativity has given an explanatory account is the puzzling **excess precession of the planet Mercury's orbit**. For half a century astronomers have known that the major axis of Mercury's orbit shifts around the plane of orbit (a rotation called precession), and its perihelion (the point of closest approach to the Sun) advances nearly 500 arc sec a century. The precession of Mercury's orbit measures 5600 seconds of arc per century. Newtonian equations are able to predict the precessions of all planets in the solar system except Mercury. Even after all the effects of other planets, the fact that Earth is not an inertial frame, and the sun's slight deformation due to its rotation have been accounted for, Newton's equations predict a precession of 5557 arc sec per century, a discrepancy between observation and prediction of 43 arc sec per century. If special relativity considerations, such as mass change and time dilations, are taken into account, the discrepancy is 21 seconds of arc per century. When general relativity is applied, the predicted motion due to curvature of spacetime yields the additional 43 arc sec per century, in line with observations to within a few percent. This is a triumph of general relativity over Newtonian mechanics.

A third test may be called **gravitational redshift**, not to be confused with the Doppler shift. General relativity predicts that clocks run slower in stronger gravitational fields than in weaker ones. Thus a clock in the basement runs slight slower than one in the attic. Consider an experiment in which a clock placed on the floor of an elevator rising from the ground. The clock emits a light pulse toward a mirror at the top of the elevator. As the elevator accelerates the mirror recedes, but eventually the light pulse reaches the mirror, where it is reflected back. On the trip up the distance covered is much greater than the height of the elevator because it has to catch up to the receding mirror. On the trip down the distance covered is shorter because the floor of the elevator accelerates upward to meet it. Calculations show that the total distance traveled by the light pulse is greater than twice the height of the elevator, which is the distance that would be covered by a clock at rest. Since the speed of light is constant, it follows that the time taken for the round trip is longer in an accelerated reference frame than in a frame at rest. By the equivalence principle, we can say that time runs slower in the presence of a gravitational field. And this is in addition to the time dilation effect. Light from a very compact star with a strong gravitational field has a higher frequency (i.e., shorter wavelength) than light coming from deep space. Thus light leaving a star with strong gravity shifts to the red (low frequency) end of the spectrum. Suppose an observer sends a light beam from a curved region of space near a massive object to another observer located in a region far away from a massive object. The distant observer will measure a lower frequency or longer wavelength for the light than the sender. We call this change of wavelength gravitational redshift. Since light has oscillations that can be used as clocks, general relativity's predictions agree with observations.

Finally, the existence of **double quasars** provides further confirmation of **gravitational lensing** predicted above. Quasars, quasi stellar radio sources, are very old, starlike objects with huge redshifts and strong radio emission, born when the universe was very young. They are some 14 billion light-years away and are receding fast. Hence, their light takes billions of years to reach us and gives us a glimpse into the early days of the universe. In 1979 astronomers Dennis Walsh, Robert Carswell, and Ray Weymann found to their surprise two quasars, called 0957+561A and 0957+561B, which are only 6 arc sec apart. Since quasars are rarely close together, they studied their emission line redshifts and found them virtually identical. They concluded that the double image came from the same source, for which an intervening, probably elliptical, galaxy acts as a gravitational lens. By 1990 nearly two dozens double quasars had been reported, of which six cases were convincing. Every new discovery of double quasars provides additional confirmation of general relativity.

All the above tests cannot be considered absolute proof of the general theory of relativity since they may be explained by other theories as well. But the validity of the principle of equivalence and the success of general relativity, whose predictions are consistent with observations, make the theory the most remarkable achievement in twentieth-century physics.

3.4 Geometry of Spacetime

As a theory of gravitation, the general theory of relativity describes gravity in terms of the geometry of spacetime. We will first examine all the possible geometries the universe can have, then discuss the basic premise of general relativity that gravity curves the fabric of spacetime. Since it is generally hard for us to visualize three-dimensional curved space, in what follows we will use examples of two-dimensional objects to illustrate their properties.

Flat or Euclidean Geometry – Zero Curvature

Euclidean geometry is familiar to us as it is taught in school. We have learned that two extended parallel lines will never meet, that the angles of a triangle total 180º, and that the circumference of a circle is equal to $2\pi r$. In this **flat geometry** there are two dimensions. Because parallel lines stretched to infinity stay apart, we call it **open geometry**. Space has infinite area, no boundary, no edge, and hence, no center, .as a center can only be determined from boundaries. Light travels in a straight line forever and never returns. In a flat universe, there is **zero curvature**, i.e., space is flat**.** Also a flat universe contains an infinite number of galaxies and hence, infinite mass. Although it is hard to imagine an infinite flat space getting larger, expansion at least of some parts is possible. Just imagine a sheet of rubber that can be stretched indefinitely. The geodesic, the shortest distance between two points, is a straight line. A traveler heading out in this space will never return to the point of departure.

Spherical Geometry – Positive Curvature

For discussion, imagine an Earth without any geological features. The surface of the Earth (or a ball) is a good example of the **spherical geometry** or **closed geometry**, which has **positive curvature**. This type of geometry is closed since parallel lines meet. While the Earth is a three-dimensional object, its surface is two-dimensional. Space in this geometry is two-dimensional, and there is neither boundary nor center. There is nowhere on the Earth its two-dimensional inhabitants can call the edge. Everywhere on the Earth is a two-dimensional surface. And the area is finite. Parallel lines if extended will eventually intersect. To satisfy yourself of this, just imagine you and your friend leaving the Equator for the North Pole, each following a line of longitude perpendicular to the Equator. Because of the curvature of the Earth you and your friend will end up meeting at the North Pole while thinking all along that you have been taking parallel paths. In this geometry, space can expand just like a balloon around its center, which is not reachable to two-dimensional inhabitants because it requires a third dimension, which is not available. The number of galaxies is finite; so is the total mass. As the ball expands all points on its surface get farther apart. There are no straight lines. The geodesic is not a straight line but an arc of a circle. Light travels along a geodesic, not along a straight line. A triangle can be drawn from three geodesics and its angles sum to more than 180º. The circumference of a small circle is still equal to $2\pi r$, but as the circles get larger, its circumference will fall short of $2\pi r$ by an increasing amount. Eventually, the circumference gets very small for a circle with a radius reaching halfway across the universe!

If this space is vast compared to its inhabitants, such as people on the Earth, its curvature is not easy to discern. However, through measuring instruments it is possible for them to describe this kind of universe with accuracy.

Hyperbolic Geometry – Negative Curvature

This geometry is best exemplified by the saddle. In **hyperbolic geometry** parallel lines diverge and hence never meet. For this reason it is called **open geometry**. The entire two-dimensional **negatively curved** space cannot be drawn, but we can approximate it by using the middle of the saddle-shape surface. The angles of a triangle on this surface total less than 180º. The circumference of a circle is always greater than $2\pi r$ though in small circles the difference is difficult to see. Like flat space, hyperbolic, negatively curved space has infinite area that extends in all directions. The number of galaxies is infinite and so is mass. It also has no boundaries and no center. Last, it can be expanded indefinitely by stretching its two-dimensional surface. When expanded this universe pushes its two-dimensional galaxies farther away from one another.

The Future of the Universe

At the present time it is not known which of the three possible geometries the universe has. Yet the future of the universe crucially depends on the type of geometry and the density of the universe.

General relativistic models use the same three models above for our three-dimensional universe although it is impossible to visualize a universe made of space that has positive, zero, or negative curvature. If the space is curved, we need an extra dimension for it to curve into. However, we do not need to imagine what a four-dimensional curved universe looks like. We can still understand its properties by extrapolating from the two-dimensional models described above.

According to a basic premise of general relativity, matter determines the spacetime curvature. Hence, the **curvature of space** and **density of matter** are interconnected. The term "matter" here refers to all sorts of matter and energy in the universe, visible as well as invisible. Since it is impossible to determine the curvature of space geometrically, scientists focus on determining the density of matter in the universe. If the average density of matter is large, the universe has positive spatial curvature; and if the density is small, the curvature is negative. When the average density of matter $\boldsymbol{\rho}$ is exactly equal to the **critical density**, symbolized by $\boldsymbol{\rho_c}$, the curvature of the universe is zero. Note that the critical density is very small, $\boldsymbol{\rho_c} = 10^{-26}$ kg/m^2 or about 10 hydrogen atoms per cubic meter. For convenience, a parameter called Omega, **Ω**, has been designed to express the ratio of the average density of matter in the universe to the critical density, $\boldsymbol{\Omega_o} = \boldsymbol{\rho} / \boldsymbol{\rho_c}$. The subscript "o" refers to present ratio. If the density of matter is less than the critical density, $\boldsymbol{\rho} < \boldsymbol{\rho_c}$, then $\boldsymbol{\Omega_o} < 1$. This means the universe is infinite with negative curvature and it will expand forever. If the average density is greater than the critical density, $\boldsymbol{\rho} > \boldsymbol{\rho_c}$, then $\boldsymbol{\Omega_o} > 1$, and the universe is finite and has positive curvature, and it will eventually contract. If the average density is equal to the critical density, $\boldsymbol{\rho} = \boldsymbol{\rho_c}$, then $\boldsymbol{\Omega_o} = 1$, the universe is infinite and flat, and it will also expand forever.

Of these three scenarios, which one is the most likely to be followed by the universe? We know that at the present time the universe is expanding. The question is whether the expansion will continue forever or whether it will eventually stop and the universe will begin to contract. Here an analogy with a tossed ball will shed light. Imagine Superman throwing a ball in the air. If the tossing force is small, at first the ball rises with appropriate acceleration against gravity, which tends to slow it. Then at some point the ball reaches its highest point, and finally it falls back down to the Earth. If the ball is thrown much more energetically, it gains tremendous speed, keeps rising forever, and never returns to Earth. The ball has reached its escape velocity. The crucial factors that determine whether the ball will return are its speed and gravity.

For the universe these two factors are the Hubble's constant, which describes the expansion of the universe, and the average density of matter, which characterizes the self-gravitation and curvature of the universe. The universe faces two alternatives. If the Hubble's constant is small and the average density is large, expansion will proceed slowly until it reaches its maximum, then the universe will stop expanding and start collapsing. This is sometimes referred to as the Big Crunch (a Big Bang in reverse), in which the universe ends in a great conflagration. Alternatively, if the Hubble's constant is large and the average density is small, the universe keeps expanding forever until all stars and galaxies have burned up all their fuel and die. We have the Big Chill. The universe will meet its fate in fire or in ice.

3. 5 Einstein's Curvature of Spacetime

The basic postulate of the general theory of relativity is that matter determines the curvature of spacetime. This means every object curves or warps the spacetime fabric around it. For a larger object the warp is more pronounced than for a smaller object. And this curving of space determines the path of any object that comes near it. Einstein does not need the gravitational force to explain the attraction that bodies exert on one another. Since all bodies follow their natural motion, spacetime curvature makes sure they follow the appropriate paths.

As an analogy of spacetime curvature, imagine a sheet of rubber stretched horizontally in mid-air. Now imagine that you place a heavy ball in the middle of the rubber sheet. It pulls the sheet down and creates a curvature in the rubber sheet. If you roll a marble on the rubber sheet from the edge, it first follows a straight path, then as it approaches the ball it moves along a curved path created in the rubber sheet by the ball. Newtonian mechanics explains this phenomenon as the effect of gravitational force exerted by the ball on the marble. In Einstein's theory, there is no force affecting the path of the marble. It simply follows the curvature of spacetime around the ball. Light too follows the spacetime curvature so that in the vicinity of a massive body light is deflected, as we have seen in light bending and gravitational lensing,

Matter not only creates a curvature in the spacetime fabric, it also slows time, as we saw in gravitational shift above.

In this and other phenomena, general relativity has provided an explanation that is both simple and elegant.

3.6 Black Holes

After a period of excitement, the general theory of relativity lost its popularity. By the late 1950's general relativity had become a dull, unproductive subject, too hard to learn and comprehend. But by the 1970's it underwent a revival and commanded renewed interest, especially in astrophysics, cosmology, and elementary particle physics. Now general relativity is front and center in the study of black holes, quasars, and in other astronomical research. We conclude this work with a brief look at the black holes.

The Formation of a Black Hole

Many stars in the Milky Way and other galaxies will die. If the star's mass is greater than 3 solar masses, it cannot become a neutron star or a white dwarf. As the star is dying, its matter burning out of control creates such intense inward pressure on all sides that it overcomes outward forces and shrink rapidly. As the sphere becomes compressed to tremendous densities, gravity on the surface increases greatly. The distortions of the spacetime fabric become more and more severe as predicted by general relativity. As the star continues contracting, light from its surface gradually bends inward and wraps around the star. Photons on its surface no longer leave the star's surface, trapped by the intense gravitational field. The escape velocity of light on its surface equals the speed of light with the result that no light escapes. The star becomes dark. At this point the space around the now diminutive star becomes so curved that it rips a hole in the fabric of the universe. The region of space around the hole looks like a funnel. The dying star is dead in this hole, invisible; it has now become a **black hole**.

A black hole is a region of spacetime from which nothing, including light, escapes.

The Structure of the Black Hole

If we approach a black hole, we first reach a region that is flat because of the weak gravitational field there. Near the hole gravity is strong, and the space curvature is pronounced. As we come closer, immediately surrounding the black hole is a sphere called **event horizon**, where the escape speed is equal to the speed of light. It is also known as the surface of the black hole.

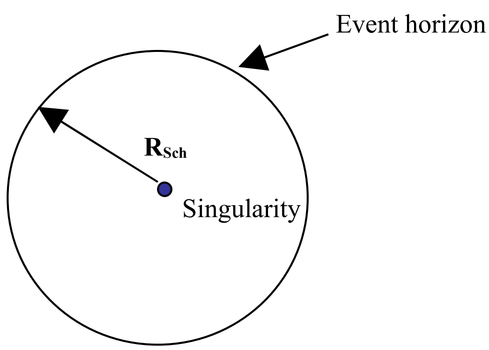


Figure 3-2. The Structure of a Black Hole

A non-rotating black hole has a singularity at its center and the event horizon surrounding it. The distance between the singularity and the event horizon is the Schwarzschild radius (R_{Sch}). Inside the event horizon the escape velocity exceeds the speed of light so that nothing in it can escape.

Once the dying star falls into the event horizon it disappears completely.

A non-rotating black hole has a simple structure, (1) the event horizon, which is its "surface" and (2) the singularity in its center. Once a dying star has imploded inside the event horizon, no force in the universe can prevent it shrinking to a point. At this stage the star has infinite density and is called a **singularity**.

The distance from the singularity to the event horizon is the **Schwarzschild radius** ($\mathbf{R_{Sch}}$), after the German astronomer Karl Schwarzschild (1873-1916), who first solved Einstein's general relativity equations. The formula for this radius is:

$$\mathbf{R_{Sch}} = \boldsymbol{GM / c^2} \qquad \textbf{(65)}$$

where G is the gravitational constant, M is the mass of the black hole. The solar mass of any object gives its Schwarzschild radius directly: The radius is 3 km for each solar mass. A black hole of 10 solar masses has a radius of 30 km.

Since gravity curves space and slows time, if an astronaut falls toward a black hole, where gravity is strong, time begins to slow down until it stops clicking in the event horizon. But the astronaut accelerates with great speed and is stretched to a thin wire by great forces. Yet to her, time ticks normally as she continues falling. The reason for the difference is that the curvature of spacetime causes time to tick more slowly for the astronaut than for an outside observer. From the astronaut's point of view she crosses the event horizon and accelerates quickly toward the singularity without noticing anything special. To the outside observer, however, she continues to slow down and finally hovers over the event horizon forever. That is not exactly correct since the light the astronaut emits becomes ever more gravitationally redshifted until she completely disappears from the observer's view.

The remarkable thing is that observer and astronaut could not agree on whether the astronaut ever crosses the event horizon.

4. Problems

4.1 Frames of Reference

4.1.1 Two airplanes A and B are flying side by side at the same altitude. Plane A is overtaking plane B at 6 km/h. A passenger in plane A is walking toward the front at 2 km/h while a flight attendant is walking toward the rear at 3 km/h. To an observer in plane B what are the speeds of the passenger and the attendant?

4.1.2 Leaving directly from the bank with the speed of 3 km/h, a boat crosses a river flowing at 4 km/h. If a passenger walks diagonally toward the boat's stern with a speed of 1.39 m/s, his path forming an angle of 53.1° toward the upstream direction from the length of the boat, what is his speed with respect to the riverbank?

4.1.3 A train passenger is running 10 km/h down the aisle against the train's velocity of 10 km/h as it moves past the station. Find (a) the passenger's speed relative to the station; (b) the passenger's speed relative to the station if he runs in the opposite direction.

4.2 Relativistic Velocity Addition

4.2.1 Two galaxies are traveling away from the earth in opposite directions, both with speed v = 0.85 *c*. What would an observer from one galaxy measure for the speed of the other galaxy?

4.2.2 A spaceship approaching Venus at a speed of 0.82 *c* launches a probe toward the planet at a speed of 0.70 *c* relative to the spaceship. What is the speed of the probe as measured by an observer on Venus?

4.2.3 A spaceship approaching the Moon at 0.5 *c* sends a light signal to an observer stationed in a lunar station. (a) What is the speed of the light signal as measured by the spaceship's pilot? (b) How fast does the light signal travel relative to the lunar observer?

4.2.4 Two spaceships leave the earth in opposite directions with the same speed of 0.50 *c* with respect to the earth. (a) What is the velocity of ship 1 relative to ship 2? (b) What is the velocity of ship 2 relative to ship 1?

4.2.5 A spaceship travels with the speed of 0.712 *c* relative to the earth A probe launched from the spaceship travels at 0.823 *c* with respect to the spaceship. What is the speed of the probe with respect to the spaceship if it is fired (a) in the direction of travel of the spaceship? (b) in the opposite direction?

4.2.6 The captain of a space station sees his rocket ship Alpha approaching with the velocity of 0.549*c*. He sends out rocket ship Beta with velocity of 0.612 *c* to meet Alpha. What is the velocity of the Alpha relative to the Beta.

4.2.7 Two asteroids head toward Venus. Asteroid Centaur approaches Venus with the speed of 0.679 *c* and Asteroid Anteus approaches Venus with the speed of 0.772 *c*. What is the speed of Centaur with respect to Anteus?

4.2.8 Starship Enterprise and Rocket Ship Intrepid are moving away from the Magnum Space Station with speeds of 0.783 *c* and 0.698 *c* respectively. What is the speed of the Enterprise relative to the Intrepid?

4.2.9 A proton moves to the left with a speed 0.789 *c* with respect to the laboratory frame. An electron moves to the right with a speed of 0.903 *c* relative to the proton. What is the speed of the electron with respect to the laboratory frame?

4.2.10 Space shuttle Valiant moves to the right with a speed of 0.55 *c* with respect to the International Space Station. Space shuttle Endurance leaves the International Space Station and heads left at a speed of 0.445 *c*. Find the speed of the Valiant relative to the Endurance.

4.2.11 For very fast speeds (but still much smaller than *c*), for example those achieved by a space shuttle, show that the relativistic velocity addition is reduced to the classical velocity addition formula.

4.2.12 Rocket ship Gamma leaves its Starbase Alpha with a speed of 0.65 *c* when it is pursued by a hostile Rocket ship named Hunter traveling with a speed of 0.72*c* relative to Starbase Alpha. The Hunter launches a missile toward Gamma with a relative speed of 0.34 *c*. How fast does the captain of Gamma see the missile approaching him?

4.2.13 Two electrons A and B have respective speeds of 0.80 *c* and .0.90 *c*. Find their relative speeds (1) if they are moving in the same direction, (2) if they are moving in opposite directions.

4.2.14 Spaceship A is flying past the earth with a speed of 0.76 *c* relative to the earth (moving to the right). Spaceship B is approaching the earth with a speed of –0.67 *c*. relative to the earth (moving left). Find the speed of ship B relative to ship A.

4.2.15 Show that if a photon traveling with velocity *c* in frame *S'* and frame *S'* is traveling with velocity *c* relative to the frame *S*, the velocity of the photon with respect to *S* is *c*.

4.2.16 A spaceship moves to the right at 0.65 *c* and a rocket ship moves to the left toward the spaceship at the speed of 0.74 *c* as measured by an Earth observer. What is the rocket ship's speed relative to the spaceship?

4.2.17 A spaceship moves away from planet Jupiter at a speed of 0.875 *c* and shoots a probe back toward Jupiter at the speed of 0.798 *c* relative to the ship. Find the speed of the probe relative to Jupiter.

4.2.18 Boat 1 crosses a stream directly at a distance *L* and returns to its destination in t_1 time. Boat 2 goes upstream a distance *L* and downstream for the same distance taking a time t_2. Compare t_1 and t_2.

4.2.19 A spaceship while moving away from Mars with a velocity of 0.80 *c* fires a rocket in the direction of the spaceship's motion, which reaches a velocity of 0.70 *c* relative to the spaceship. What is the rocket's speed relative to Mars?

4.2.20 A rocket ship travels to a star 15 x 10^{18} m/s away with a speed of 1.275 x 10^{8} m/s as measured by an observer at the launch site. How long will the trip take according to an onboard clock?

4.2.21 Two galaxies are traveling away from the earth in opposite directions with the same speed of 0.85 *c*. What would the speed of each galaxy be as measured by an observer in the other?

4.2.22 If a quasar moving away from Earth with speed of 0.75 *c* ejects matter toward Earth with a speed of 0.45 *c*, what is the speed of the ejected material as measured from Earth?

4.2.23 In a laboratory experiment an electron moving at 0.8995 *c* collides with a positron traveling at 0.9895 *c*. At what speed do the particles approach each other?

4.2.24 A spaceship is traveling towards a point on Earth when a laser beam from the same point fires toward the ship. At what speed does the laser beam travel as measured by an engineer aboard the ship who receives it? Resolve the problem classically.

4.2.25 Two light pulses are sent out in opposite directions. What is the speed of either one with respect to the other? Resolve the problem classically.

4.2.26 An earthbound observer finds that two spaceships traveling at the speed of 0.825 *c* are moving in opposite directions, one toward Earth, the other away from Earth. What is the speed of one spaceship with respect to the other?

4.2.27 Two spaceships are racing toward Earth. Ship A in the lead moves at 0.89 *c* and separates from ship B at 0.54 *c*. (a) With respect to Earth what is ship B's speed? (b) Calculate the relative speed between the two ships if ship A increases its speed by 5%.

4.2.28 Asteroid A and asteroid B are receding from Earth with speeds relative to Earth of 2 x 10^{8} m/s and 2.5 x 10^{8} m/s, respectively. What is their speed relative to each other?

4.2.29 A probe is launched in the direction of motion of its spaceship with a speed of 1 x 10^{8} m/s. If the spaceship's speed relative to Earth is 5 x 10^{8} m/s, what is the probe's speed relative to Earth?

4.2.30 Two particles fly in opposite directions, each with the speed of 0.77 *c*. Find the relative speed of the particles.

4.3 Lorentz Transformation and Simultaneity

4.3.1 At time $t = t' = 0$, the origins of frame *S* and *S'* overlap. *S'* moves at $v = 1.2$ x 10^{8} m/s with respect to *S*. A particle in *S'* comes to a rest at coordinates $x' = 35$m, $y' = 40$m, and $z' = 0$. What are its coordinates in frame *S* at (a) $t = 3.5$ μs? (a) $t = 45.5$ μs?

4.3.2 Suppose two inertial systems *S* and *S'* have the same *x*-axis and parallel *y*-axes. Frame *S'* moves with a speed of 0.65 *c* relative to *S* in the +*x*-direction. A rocket in *S* moves with a speed of 0.2 *c* in the +*y*-direction. (a) What does an observer in *S'* calculate the speed of the rocket to be? (b) What is the direction of the rocket as seen by that observer?

4.3.3 Show that at everyday velocity, $v << c$, the Lorentz transformation reduces to the Galilean transformation.

4.3.4 A rocket ship moving away from Earth with a speed of 0.824 c fires a probe in the direction of its motion with the speed of 0.56 c relative to the ship. What is the probe's speed with respect to Earth?

4.3.5 Show that (a) the Lorentz transformation reduces to the Galilean transformation if $v = 0$. (b) the Lorentz transformations predict length contraction of a moving object.

4.3.6 At the beginning of a spaceship's interstellar voyage to a star 100 ly away, $x = x' = 0$ m and $t = t' = 0$ s. Its speed as measured by observers at its station is 0.500 c. Find (a) the distance x and the time t as measured by home station's astronauts when the spaceship arrives at the star, and (b) the spaceship's x' and t' at its destination as measured by onboard astronauts.

4.3.7 Two lights are turned on simultaneously at the moment fast-moving Observer 1 is passing stationary Observer 2. (a) Do these lights appear to flash simultaneously for either observer or both observers? (b) Adduce evidence to support your answer from a Lorentz transformation expression.

4.3.8 A stationary observer sees two fireworks explode one 5.00 seconds after the other. What does an observer moving at a speed of 0.55 c past the fireworks measure the time interval between the two explosions to be?

4.3.9 Moving past Earth at velocity 2.5 x 10^8 m/s a spaceship beams laser signals to Earth every 12 seconds according to an onboard clock. What does a clock on Earth record this time interval to be?

4.3.10 At the common origin of two frames S and S' a light pulse is emitted at time $t' = 0$. At time t its distance from the origin is $x'^2 = c^2t'^2$. By using the Lorentz transformation, show that the motion is exactly the same in both frame S and S', i.e., that the distance in frame S is $x^2 = c^2t^2$.

4.4 Michelson-Morley Experiment

4.4.1 Starting from the same airport two airplanes fly for 400 km at the same airspeed of 300 m/s, one heading east for Pittsburgh, the other heading north for Detroit. The wind blows at 50 km/h in the easterly direction. Find: (a) the time of flight to each city, (b) the time of the return trip, and (c) the difference of the total flight times between to two airplanes. *Hint:* Use the boat analogy in Figure 2-2.

4.4.2 In the Michelson-Morley experiment, beam 1 travels across the ether wind and makes the round trip in time $t_1 = 2\, l / c\ (1 / \ (1 - v^2 / c^2)^{1/2})$ while beam 2 travels with and against the ether wind making the round trip in time $t_2 = 2\, l / c\ [1 / \ (1 - v^2 / c^2)]$. Suppose the M_s - M_2 arm shrinks as suggested by the Lorentz-FitzGerald contraction. Show that $\Delta t = t_2 - t_1 = 0$.

4.4.3 In one run of the Michelson-Morley experiment, the perpendicular arms of the apparatus have length L = 25 m. Given v = 3 x 10^4 m/s, calculate (a) the time difference caused by the rotation of the interferometer, and (b) the expected fringe shift, if the light beam used has a wavelength of 540 nm.

4.4.4 Find the γ associated with a particle moving at the speed of (a) 0.005 c; (b) 0.15 c, (c) 0.00005 c, and (d) 0.6675 c.

4.5 Time Dilation

4.5.1 Captain Aurore of Spaceship Longtrek embarks on a round-trip voyage to a faraway Galaxy Tresloin situated 15.5 ly from Earth traveling with a velocity of 0.75 c. When he comes back to Earth how old is Captain Aurore if he left at age 27?
Hint: Use $v = d / \Delta t$.

4.5.2 Commander Desbois left his 8-year-old daughter on a journey to star Diana traveling with a velocity of 0.85 c. The round trip took him 6.84 years as measured from Earth. How old is his daughter when he comes home?

4.5.3 Captain Agamemnon took a 25-year round trip to Nebula Troy and turned back. When he came home, his wife Clytemnestra found him to be 7 years older. What was the average speed of his spaceship during the trip?

4.5.4 Hurtling at velocity 0.7885 c toward star Labas far away, spaceship Intrepid reaches its destination and heads back to Earth. On his return, Captain Achilles' wife found him 8 years older than when he had left. How many years did Captain Achilles' round trip last from his wife's point of view?

4.5.5 Expedition Leader Hercules spent 17 years of his starship life in search of the Heavenly Golden Fleece. On his return to Earth, 2000 years had passed judging by the historical records kept at the Library of World Congress. What was the average velocity of his voyage?

4.5.6 In the parallel universe PU2 all the physical constants are different from the physical constants of our universe. Suppose a physicist in PU2 standing on the platform clocks a passing train as traveling with the speed of 200 km/s. His colleague on the train finds the elapsed time of a flash of light from the train's floor to its mirrored ceiling and back to be 65 s while he recorded the same two events as lasting 59 s. What is the speed of light in PU2?
Hint: What was anomalous about this scenario?

4.5.7 In the parallel universe PU3 all the physical constants are different from the physical constants of our universe. Suppose a physicist in PU3 standing on the platform clocks a passing train as traveling with the speed of 2000 km/s. His colleague on the train finds the elapsed time of a flash of light from the train's floor to its mirrored ceiling and back to be 75 s while he recorded the same two events as lasting 89 s. What is the speed of light in PU3?

4.5.8 A cosmonaut's trip to a distant star takes 11.783 years traveling with a speed of 0.992 *c* relative to Earth. How many years does the cosmonaut age during this trip?

4.5.9 An astronaut spends 20 minutes eating lunch by his spaceship's time. Assuming the astronaut is traveling at 0.887 *c* relative to Earth, (a) how long does the astronaut's lunch last from Earth's reference frame? (b) how far does the spaceship travel during this time from Earth's point of view?

4.5.10 How much have you aged flying the distance of 8,500 km from Houston to Paris in a jet airliner traveling with the speed of 1000 km/h compared to when you left Houston?

4.5.11 Planning a trip to Andromeda galaxy 2 million ly away, you figure it will take 34 years by a clock on the spaceship. How fast will the spaceship have to travel on average to complete the planned trip?

4.5.12 On a round trip to a galaxy 30 ly away 42-year-old Astronaut Jouvence travels with a speed of 0.99 *c*. His son Hippolyte was 19 when he left. (a) How old is Jouvence on his return to Earth? (b) How old is his son Hippolyte when he sees his father again?

4.5.13 Mission Control on Earth measures Astronaut Irina's heartbeat to be 80 beats per minute. She is traveling with a velocity of 0.65 *c* relative to Earth. What does she measure her heart rate to be?

4.5.14 A jet plane flying with a constant velocity of 330 m/s requires 3 h by the onboard clock to reach its destination. What is the duration of the trip by the clocks on the ground?

4.5.15 The lifetime of a neutron as measured on Earth is 6.42×10^2 s. What is the lifetime of the neutron as measured by a clock on a spaceship flying past at a velocity of 2×10^8 m/s?

4.5.16 As spaceship Intrepid cruises at a constant speed of 0.894 *c* past planet Diana, its captain Goddard observes that the Diana clock and his own clock both read 2:00 p.m. At 3:30 p.m. by his clock, Captain Goddard's ship passes by planet Athena. If the Diana and Athena clocks are synchronized with each other, what time do Athena's clocks show when spaceship Intrepid passes by?

4.5.17 Captain Hubble wants to make a trip to a star 35 ly away. How fast does he have to travel to make the trip in only 8 years?

4.5.18 What is the mean life of a muon falling down with a speed of 0.9999 *c* relative to Earth if its mean life at rest is 2.2 μs?

4.5.19 The unstable muon decays in 2.3 μs at rest. (a) Find the muon's mean life in a laboratory if it travels with a speed of 0.99 *c* with respect to the lab. (b) How far has the muon particle traveled during its lifetime as measured in the lab?

4.5.20 A certain particle, which decays after 16 μs, has traveled at distance of 5500 m. What is its mean velocity during is lifetime?

4.5.21 After a 40-year intergalactic voyage, as measured by his younger brother on Earth, Captain Hawking came home to find he is 5 years older than when he left, and his younger brother twice as old as he is. Find (a) the average speed he achieved during his journey. (b) his brother's age on his departure and his age after the voyage if their total age at reunion is 75 years.

4.5.22 Given a π^+ meson's mean life of 2.5×10^{-8} s, if it travels with a velocity of 0.993 *c*, (a) what is the distance it covers before decay? (b) Suppose there were no time dilation effects, what would this distance be?

4.5.23 The average life of a neutron, as measured on Earth, is 6.45×10^2 s. What is the mean life of this neutron if measured by an observer on a spaceship traveling past with a speed of 2.5×10^8 m/s?

4.5.24 π-mesons, subatomic particles produced in particle physics experiments and in cosmic radiation, have a lifetime of 0.84×10^{-16} s in low-energy debris of a particle physics experiment and a mean lifetime of 1.4×10^{-16} s as cosmic rays hitting our atmosphere. (a) Give a reason for the discrepancy in these lifetimes. (b) Calculate the γ and v of each type of meson.

4.5.25 If an object moves at a speed $v = 1 / 4\ c$ as measured by a stationary observer, find its γ.

4.5.26 What is the speed of a muon, which has a mean life at rest of 2.2 μs, if its mean life in motion is 5.9 μs?

4.5.27 An onboard atomic clock records 42 hours on the around-the-world trip made by a jet plane flying at the speed of 1000 km/h. What is the difference between this time and the time measured by an identical and synchronized atomic clock in the laboratory on Earth?

4.5.28 If a space-faring astronaut wants to come back to Earth only 5 years older by her clock than when she left, at what speed must she travel on a 30-ly round trip?

4.5.29 How far does a particle travel after reaching its destination in 87.2 μs from its rest position where it decays in 4 μs? What is its speed of travel?

4.5.30 A jet airplane flying with a speed of 500 m/s carries an atomic clock, which measures a time interval of 3500 s. What does an identical clock in a laboratory on the ground measure the same interval to be?

4.6 Length Contraction

4.6.1 The distance between two planets is measured by an Earth observer to be 987,000 km. If a spacecraft takes 2 seconds by the clocks on the ship to fly with a uniform velocity from one planet to the other, what is its average speed?

4.6.2 What is the Lorentz contraction of a car traveling at 150 km/h?

4.6.3 As a 150-meter-long rocket moves away from Earth at a constant velocity, a light pulse is beamed to its nose and tail, and is reflected back to Earth. The reflected signal from the tail reaches a detector on Earth 250 s after emission, and the signal reflected from the nose follows 1.5×10^{-6} s later. Find the distance between the rocket and the earth and the rocket's velocity relative to Earth.

4.6.4 Moving past a meter stick at a speed of 0.55 c, an observer measures its length. What length does the observer measure the meter stick to be?

4.6.5 Calculate the length of a meter stick that moves along its length with a speed of 2.89×10^8 m/s?

4.6.6 A rocket ship moves at the speed of 0.45 c to an earth observer. (a) Find its γ. (b) Find its contracted length L as observed by the earth observer. (c) Find the ΔL, the change in the ship's length in the direction of motion as observed by the same observer. (d) Find its ΔH, the change in its height in the direction perpendicular to its motion, as observed by the earth observer. (e) Find its contracted length L as observed by an onboard astronaut.

4.6.7 The space shuttle is 37.23 m on the runway. As it hurtles at 27,358 km/h into space, how long does it appear to an observer on Earth?

4.6.8 A meter stick on a spaceship measures 0.75 m. How fast does the spaceship travel relative to an Earth observer?

4.6.9 Captain Highspeed travels by you in a 25-m-long spaceship at a very high speed. You are traveling in a spaceship that measures 20 meters on the ground. From your point of view, (a) how long is Captain Highspeed's spaceship, (b) how long is your ship in flight, and (c) how fast is Captain Highspeed's spaceship?

4.6.10 A 3-m-long car travels at a speed of 120 mi/h. What is the decrease in the car's length as measured by a stationary observer?

4.6.11 How fast must a rocket ship travel so that its length shrinks by two-thirds to an observer at rest on the earth?

4.6.12 How long is a spaceship at rest if an observer on the earth measures its length to be 95 m as it flies by the observer at the speed of 0.75 c?

4.6.13 To make a trip to a star 45 ly away, how fast would an astronaut have to travel to make the trip in 4 years?

4.6.14 To make a trip to a star 50 ly away, how fast would an astronaut have to travel so that the distance would be only 15 ly?

4.6.15 Sitting in your Batmobile you see your friend pass you in her 8-m-long Batmobile at a speed of 0.12 c. Your friend says she measures your car to be 8.5 m long. What do you measure for the lengths of the two cars?

4.6.16 At what speed would an astronaut have to travel to reach the closest star Alpha Centauri 4.0 ly from Earth in only 2.5 years as measured by a clock in the spaceship? At that speed how far did the astronaut travel in his reference frame? What would an earthbound observer measure for the time the astronaut spent to reach the star?

4.6.17 A spaceship 30 m in height and 200 m in length travels past an observer on Earth at a speed of 0.875 c. (a) What does the observer measure for the height and length of the spaceship? (b) If the observer measures an experiment on the earth to last for 25 seconds on her watch, how much time did she find the same experiment to last as measured by the spaceship's clock? (c) How fast did the observer appear to be moving according to the astronaut? (d) How many seconds did the astronaut see elapse on the observer's clock when 25 seconds passed on hers?

4.6.18 In a laboratory experiment an electron travels 4 cm in 0.25 ns. (a) What is the electron's speed? (b) In the reference frame of the electron, what was the distance the lab traveled?

4.6.19 Your spaceship and your friend's spaceship are traveling toward a galaxy. Your speed is 0.75 *c* relative to your friend's. Your friend measures the lengths of both spaceships and finds them to be the same. (a) If you measure the length of your friend's ship, do you find your ship longer or shorter than her ship's length? Explain. (b) Calculate the ratio of the length of your ship to the length of your friend's.

4.6.20 Captain Kirk and Captain Cousteau travel through space in identical spaceships. After making measurements Captain Kirk finds his ship $8/7^{th}$ times as long as of Captain Cousteau's ship. (a) From Captain Cousteau's point of view, how long is Captain Kirk's ship if the ships at rest is 120m? (b) What is Captain Kirk's speed relative to Captain Cousteau's?

4.6.21 On a trip to a star an astronaut moving with the speed of 0.8 *c* relative to Earth finds the distance covered from her reference frame to be 7 ly from Earth. On the return trip, the astronaut travels at a speed of 0.97 *c* relative to Earth. What is the distance in light-years the astronaut traveled on the return trip as measured by the astronaut?

4.6.22 A laboratory tube in which protons move at a mean speed of 0.5 *c* measures 70 m long. In the protons' reference frame, how long does the tube measure?

4.6.23 A rectangular spaceship 25 m high and 140 m long is traveling parallel to its length. What is the speed of the ship when an earth observer finds the ship to be a square?

4.6.24 The height and length of a triangular spaceship measure 30 m and 100 m respectively. At what speed must it travel in a direction parallel to its length for its length-to-height ratio to be 5/3?

4.6.25 An astronaut traveling to a star 8 ly away with a speed such that the gamma factor is $\gamma = 7/6$. What is the distance from Earth to the star as measured by an earthbound instrument?

4.6.26 If a 50-m-long rocket flying past a stationary observer appears to measure only half its original length, what is its speed?

4.6.27 How fast does a sports car have to travel in order for a stationary observer to find its length to be shortened by 10% compared to its length at rest?

4.6.28 An electron travels a straight distance of 15 m as measured in its own frame of reference. An experimenter finds the electron's speed in the particle accelerator to be 0.999999 *c*. What is the distance traveled by the electron as measured by the experimenter?

4.6.29 A supersonic jet plane measuring 100 m in length on the runway flies at the speed of 3000 km/h. (a) What is the difference in the lengths of the jet plane when it is at rest and when it is flying as observed by an air traffic controller? (b) How fast does the plane have to travel for this difference to quadruple?

4.6.30 Event A and event B occur in frame *S* at the same point. Event B occurs 3 s after event A. In frame *S'* moving relative to *S* event B occurs 5 s after event A. What is the distance between the positions of the two events as measured in frame *S'*?

4.7 Relativistic Momentum

4.7.1 Calculate the momentum of a proton moving with the speed of (a).0.10 *c*, (b) 0.67 *c*, (c) 0.75 *c*, and (d) 0.95 *c*. Calculate the momentum of an electron moving with the same speeds.

4.7.2 If an electron's relativistic momentum is 95% larger than its classical momentum, (a) what is its speed? (b) If the particle were a proton, what would be the proton's speed?

4.7.3 To achieve the same momentum as an electron moving at a speed of 0.95 *c*, at what speed must the proton move?

4.7.4 At what speed must an object travel so that its relativistic momentum is five times greater than its classical momentum?

4.7.5 A rocket ship with a mass of 5×10^8-kg moves away from Mars with a speed of 0.75 *c*. What are its classical momentum and its relativistic momentum?

4.7.6 Find the relativistic momentum of a proton with a mass of 1.673×10^{-27} kg moving at speed of 2/3 *c*.

4.7.7 A neutron moves with speed $v = 0.75\ c$ and has a mass of 1.675×10^{-27} kg. Determine its relativistic momentum.

4.7.8 If an electron has a momentum 85% larger than its classical momentum, how fast is it moving?

4.7.9 If a proton has 1836 times the mass of an electron, calculate the speed at which the electron will have the same relativistic momentum as the proton moving at 0.020 *c*.

4.7.10 Calculate the speed of an object whose relativistic momentum is five times greater than its classical momentum.

4.7.11 Calculate the speed at which a proton has a momentum equal to that of an electron moving at speed of 0.75 *c*.

4.7.12 By how much is classical momentum in error vis-à-vis relativistic momentum if an electron with mass 9.11 x 10^{-31} moves at speed of (a) 0.1 *c*, (b) 0.45 *c*, (c) 0.91 *c*?

4.7.13 What is the ratio of relativistic momentum to classical momentum for a muon traveling at 0.95 *c*?

4.7.14 What is the speed in terms of *c* of a muon such that its relativistic momentum is 25% greater than its nonrelativistic momentum?

4.7.15 For an alpha particle with mass of 6.644 x 10^{-27} kg to have the same momentum as a neutron with mass of 1.674 x 10^{-27} kg traveling at speed of $v = 0.92\ c$, how fast must the alpha particle travel?

4.7.16 How fast must a particle travel for its mass to be five times its rest mass?

4.7.17 A neutron with mass 1.674 x 10^{-27} kg has a momentum 2.8 x 10^{-19} kg.m/s. What is its speed?

4.7.18 At what speed does the momentum of an alpha particle becomes three times as great as its classical momentum?

4.7.19 Calculate the classical and relativistic momenta of a 3.4-kg mass moving at 0.89 *c*.

4.7.20 An electron with a mass of 9.11 x 10^{-31} kg moves with speed $v = 0.77\ c$. Find its classical and relativistic momenta.

4.8 Relativistic Mass

4.8.1 How much does a jet airliner increase in mass if it flies at 1000 km/h?

4.8.2 If an electron moves with a speed of 0.50 *c*, what is its relativistic mass?

4.8.3 A moving particle doubles its mass with respect to its rest mass. How fast does the particle move?

4.8.4 To have a relativistic mass of 0.5 kg, how fast must a 0.20-kg baseball be thrown?

4.8.5 After a collision with an asteroid at rest a rocket with a mass of 6.7 x 10^{7} kg traveling at a speed of 0.6 *c* remains embedded in the asteroid. Calculate the asteroid's mass if the rocket-asteroid system has the speed of 0.45 *c*.

4.8.6 What is the mass of a proton moving at 0.65 *c*?

4.8.7 What would be the percent increase in mass of a 4.5 x 10^{6} kg space robot traveling at 40000 km/h?

4.8.8 How fast must an object move in order for its mass to be 2 percent greater than its rest mass?

4.8.9 What is the mass of a proton that is traveling at $v = 0.92\ c$?

4.8.10 At what speed must a 0.20-kg baseball be thrown to have a mass of 0.5 kg?

4.8.11 How much mass is converted to energy per hour in a nuclear reactor that produces 15 x 10^{10} W of power?

4.8.12 How much mass does the sun burn every second to radiate energy at the rate of 3.92 x 10^{26} W? Assuming a mass of 1.99 x 10^{30} how long will the sun continue to radiate energy before all its mass is totally consumed?

4.8.13 What is the mass while in flight of a 0.5-kg ball thrown at $v = 0.65\ c$?

4.8.14 Accelerated to 0.93 *c* along a straight path in an accelerator, an electron has an increased mass 10000 times its rest mass when it reaches its final speed. What is the electron's final speed?

4.8.15 At what speed will the mass of an electron be 0.825 MeV?

4.8.16 At what speed will the mass of an object be six times its rest mass?

4.8.17 If all the mass of an object is converted into energy and the energy observed is 9.123 x 10^{16} J, what is the object's mass?

4.8.18 An energy of 60 MeV is generated with all the mass of a certain particle. What was its mass?

4.8.19 How much energy in J and MeV can a 5-g coin produce assuming all its mass is used up in the process?

4.8.20 A future rocket ship is capable of converting chickens into energy. Assuming it carries 100 3-kg chickens on board, how much energy in J and MeV can the chickens generate?

4.9 Relativistic Energy

4.9.1 Suppose an electron's kinetic energy is 2% larger than the value that would be measured had the relativistic effects not existed. What is the electron's speed?

4.9.2 An electron of mass 9.55 x 10^{-31} kg has a speed of 0.895 *c*. Calculate: (a) the electron's total relativistic energy in Joules and MeV. (b) its total relativistic kinetic energy in Joules and MeV. (c) its relativistic momentum. (d) its classical kinetic energy and momentum.

4.9.3 Calculate the ratio of relativistic kinetic energy to classical kinetic energy of a particle traveling at velocity = 0.85 *c*.

4.9.4 At what speed does a traveling particle's rest energy equal its relativistic kinetic energy?

4.9.5 Suppose you could convert all the mass of 3-g penny into energy. What is the energy produced in J and in MeV?

4.9.6 Assuming you want to produce a total energy of 2000 MeV from a proton. Find its speed, momentum, and kinetic energy.

4.9.7 A neutral pion at rest decays into two oppositely directed gamma-ray photons. Find the energy and momentum of the photons.

4.9.8 Calculate the rest energy of an electron whose mass is 7 times its rest mass.

4.9.9 Find the rest energy of a neutron in J and in MeV.

4.9.10 What is the speed and momentum of a proton whose kinetic energy is 2/3 its rest energy?

4.9.11 A nuclear reactor produces an output of 1.25×10^3 MW of power a year. What is the change in mass in its reactor fuel, assuming that all the energy released by fission is converted to electricity? (In reality, only a small fraction of the energy is converted)

4.9.12 An object is traveling at a speed such that $\beta = 0.60$ and $\gamma = 1.2$. Find its total energy if its mass is 1.5×10^{-2} g.

4.9.13 Find the total energy, the momentum, and the speed of an electron with a rest energy of 0.511 MeV and a $\gamma = 1.75$.

4.9.14 Calculate the total energy, the rest energy, the kinetic energy, and the momentum of a particle with a mass of 1.5×10^{-6} kg traveling with speed of 0.35 *c*.

4.9.15 Suppose the total energy needed is 12×10^{20} J. How much mass annihilation would occur in nuclear reaction to supply that energy?

4.9.16 If an electron's kinetic energy is four times its rest energy, find (a) its total energy in MeV and in J, (b) its speed in terms of *c*.

4.9.17 Find an electron's total energy MeV and in J, its kinetic energy MeV and in J, and its momentum if it moves with speed 0.75 c.

4.9.18 A proton has a kinetic energy equal to 2/3 its rest energy. Find the proton's momentum and speed.

4.9.19 Calculate the kinetic energy and momentum of a neutron with a mass of 1.675×10^{-27} kg traveling 10×10^7 m/s.

4.9.20 If a spaceship with a mass of 5.2×10^7 kg has a relativistic kinetic energy of 4.3×10^7 J, find its speed.

4.10 Relativistic Doppler Effect

4.10.1 A green light from a star has a frequency 5.5×10^{14} Hz measured in its rest frame. What frequency does Captain Eteocles observe for the star if his spaceship travels toward the star with the speed of 0.20 *c*?

4.10.2 A certain star has a measured wavelength of 486.112 nm. According to the records in the laboratory this star has a wavelength of 486.133 nm. Is the star moving toward or away from us? What is its speed?

4.10.3 If a quasar has a redshift of 0.16, what is its radial velocity? What is its distance?

4.10.4 An astronomer studying the red light of hydrogen emitted from a star measures its wavelength to be longer than the wavelength of 658 nm measured in the laboratory. Is the star moving toward or away from Earth? Suppose the star's speed is 30000 km/s, what does the astronomer measure for the wavelength of the light?

4.10.5 On board a spaceship moving at 0.3 *c* toward Polynices Star radio signals from the star are received at a frequency of 106 MHz. On what frequency are the radio signals received on the star?

4.10.6 If a certain galaxy which emits an orange light with a frequency of 5.100×10^{14} Hz is receding from Earth with speed 3100 km/h, what is the light's frequency as observed by an earthbound observer?

4.10.7 Two spaceships are approaching each other head-on. The distance between them is diminishing at the rate of 950 km/s. Ship 1 sends a laser beam toward Ship 2. What was the wavelength of the laser at transmission if it has a wavelength of 635 nm when received by Ship 2?

4.10.8 What is the frequency of a light signal sent by Space Station Creon to the approaching supply ship Antigone traveling at 0.3 *c* when the signal received has a wavelength of 645 nm?

4.10.9 Find the wavelength of a light signal with an observed frequency of 5.95×10^{14} Hz if the source is moving away at the speed of 0.25 *c*.

4.10.10 A star is observed to have a redshift of 0.35. How fast is it moving? What is its distance at the time of observation?

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